¶ **1.** If U is an open set and F is closed set in a topological space $(X, \mathcal{T})$, then $U \setminus F$ is open and $F \setminus U$ is closed. (Recall that if $A, B \subset Y$, then $A \setminus B = \{y \in Y \mid (y \in A) \,\&\, (y \notin B)\}$.)

Solution. Note that $U \setminus F = U \cap (X \setminus F)$. If F is closed, then $X \setminus F$ is open and thus $U \setminus F$ is also open. □

¶ **2.** Let X be the real numbers with the topology which has the sets (a, ∞), $a \in \mathbf{R}$, as a base. Which sequences converge to which points? What is the closure of $(-\infty, 0)$?

Solution. The open sets are the sets (a, ∞), $\emptyset$ and $\mathbf{R}$, because the union of any family of sets of the form (a, ∞) is a set of the form (a, ∞) (or $\emptyset$, or $\mathbf{R}$). From this it follows that $(-\infty, 0)$ is closed.
A set N is a neighborhood of x if there is $\epsilon > 0$ such that $(x - \epsilon, \infty) \subset N$. Hence, a sequence x_n converges to x if and only if for any $\epsilon > 0$ there is $n(\epsilon)$ such that $x_n + \epsilon > x$ for all $n \geq n(\epsilon)$. Using concepts from Math 350, this condition is equivalent to saying that $\liminf x_n \geq x$. □

¶ **3.** (Sorgenfrey line) Show that the sets $[a, b)$, a, b real numbers, form a base for a topology on the real line. Determine which of the following subsets of X are open and which ones are closed: $(-\infty, a)$, $[a, b)$, $[a, \infty)$, (a, b), (a, ∞), $(-\infty, a]$, $[a, b]$, $\{a\}$.

Solution. $(-\infty, a) = \bigcup_{r>0}[a - r, a)$ is open because it a union of open sets. It is also closed because its complement $\mathbf{R} \setminus (-\infty, a) = [a, \infty) = \bigcup_{r>0}[a, a + r)$ is a union of open sets. □

¶ **4.** Let X be the set of integers with the cofinite topology: $U \subset X$ is open if $X \setminus U$ is finite or equal to X. Prove that the sequence $\{1, 2, 3, \cdots\}$ converges to all points of X. What are the convergent sequences in X.

Solution. Let $x \in X$. If W is a neighborhood of x, then $X \setminus W$ is a finite subset of X that does not contain x. If $n_W = \max\{n \mid n \in X \setminus W\}$, then $x_n = n \in W$ for all $n > n_W$. □

¶ **5.** Let X be a set and let $\mathcal{T}$ be the collection of all subsets U in X such that $X \setminus U$ is at most countable or equal to X.

(a) Prove that $\mathcal{T}$ is a topology on X.

(b) Describe all convergent sequences in X.

(c) Let X be the set of real numbers. Find a subset S of X and a point in the closure of S that is not the limit of any sequence of points in S.

Solution. (a) If X is countable, this is the discreet topology.

(b) A sequence $x_1, x_2, x_3, \cdots$ converges to a point x if and only if there is N such that $x_n = x$ for all $n > N$.

Indeed, suppose that (x_n) converges to x but that for every$N > 0$ there is $n > N$ such that $x_n \neq x$. This implies that there is a subsequence (x_{n_k}) of $(x_n)$with all $x_{n_k} \neq x$. The open set $U = X \setminus \{x_{n_k} \mid k = 1, 2, \cdots\}$ contains x and witnesses the failure of (x_n) to converge to x.

□

¶ **6.** Let X be the set of positive integers $n \geq 2$. Show that the sets $U_n = \{x \in X; x \,\text{divides}\, n\}$, $n \geq 2$, form a base for a topology on X. Find the closure of the one-point sets $\{x\}$, $x \in X$, and of the set of prime numbers.

¶ **7.** For each positive integer n, let $S_n = \{n, n + 1, \ldots\}$. The collection of all subsets of $\mathbf{N}$ which contain some S_n is a base for a topology on $\mathbf{N}$.

¶ **8.** Prove or disprove:

(a) The intersection of an arbitrary family of topologies on X is a topology on X.

(b) The union of two topologies on X is a topology on X.

Solution. (a) True (b) False □

¶ **9.** If $\mathfrak{B}$ is a base for a topology on X, then the topology generated by $\mathfrak{B}$ equals the intersection of all topologies on X that contain $\mathfrak{B}$.

¶ **10.** Let X be a set. A closure operator κ on X is a mapping that to each subset A of X assigns another subset A^κ of X and such that it satisfies the following properties:

(a) $\emptyset^\kappa = \emptyset$.

(b) $A \subset A^\kappa$ for all $A \subset X$.

(c) $(A^\kappa)^\kappa = A^\kappa$, for all $A \subset X$.

(d) $(A \cup B)^\kappa = A^\kappa \cup B^\kappa$, for all $A, B \subset X$.

Prove that if κ is a closure operator on X, then there is a unique topology $\mathcal{T}$ on X so that the closure of A in $(X, \mathcal{T})$ is A^κ, for all $A \subset X$.

Solution. Let C denote the family of all subsets A of X such that $A^\kappa = \kappa$. It will be shown that C is the collection of closed subsets for a topology on X. Because of (a), $\emptyset \in C$. Because of (b), $X^\kappa = X$, so $X \in C$. Because of (d), if A and B are in C, then $(A \cup B)^\kappa = A^\kappa \cup B^\kappa = A \cup B$, so $A \cup B$ is also in C.
Note that if $A \subset B$, then $A^\kappa \subset B^\kappa$. Indeed $B = (B \setminus A) \cup A$, and thus, by (d), $B^\kappa = ((B \setminus A) \cup A)^\kappa = (B \setminus A)^\kappa \cup A^\kappa \supset A^\kappa$.
Let $\mathcal{F} \subset C$ be a family of elements in C. If $\mathcal{F}$ is the empty family, then $\bigcap_{A \in \mathcal{F}} A = X$, which is in C. If $\mathcal{F}$ is non-empty, then $\bigcap_{A \in \mathcal{F}} A \subset A$, for any A in $\mathcal{F}$, and so $\left(\bigcap_{A \in \mathcal{F}} A\right)^\kappa \subset A^\kappa = A$ by the note above. In combination with (b), this implies that $\left(\bigcap_{A \in \mathcal{F}} A\right)^\kappa = \bigcap_{A \in \mathcal{F}} A$.
Let now $\mathcal{T}$ denote the topology on X determined by the family of closed sets C (U is in $\mathcal{T}$ if and only if $X \setminus U$ is in C. Let A^- denote the closure of a set $A \subset X$ in the topology $\mathcal{T}$: B^- is the intersection of all the closed subsets of X that contain B, equivalently, $B^- = \bigcap_{A \in C, A \supset B} A$. If A is in C then $A^\kappa = A$, and if $A \supset B$, then $A \supset B^\kappa$. Therefore, $B^- \subset B^\kappa$. Because (c), $(B^\kappa)^\kappa = B^\kappa$, so B^κ is in C, and because of (b), $B \subset B^\kappa$. This implies that B^κ is a closed subset of $\mathcal{T}$ which contains B, and so $B^- \subset B^\kappa$. □