13 Sequences of functions

Definition 13.1. A sequence of functions f_n converges pointwise to a function f on a set S (contained in the domain of all f_n) if for every x in S, the sequence $f_n(x)$ converges to $f(x)$. That is, for any x in S, for any $\varepsilon > 0$, there is a natural number N such that, if $n > N$, then $|f_n(x) - f(x)| < \varepsilon$.

Definition 13.2. A sequence of real valued functions f_n converges uniformly to a function f on a set S if for every $\varepsilon > 0$ there is a natural number N such that if $n > N$, then $|f_n(x) - f(x)| < \varepsilon$ for all x in S.

Theorem 13.1. *If (f_n) converges uniformly to f on S and each f_n is continuous on S, then f is continuous on S.*

Proof. Let x_0 be a point in S. Given $\varepsilon > 0$, choose N such that $|f_n(x) - f(x)| < \varepsilon/3$ for all x in S and all $n > N$. Since f_{N+1} is continuous at x_0, there is $\delta > 0$ such that, if $|x - x_0| < \delta$, then $|f_{N+1}(x_0) - f_{N+1}(x)| < \varepsilon/3$, Then, if $|x - x_0| < \delta$, $|f(x_0) - f(x)| \leq |f(x_0) - f_{N+1}(x_0)| + |f_{N+1}(x_0) - f_{N+1}(x)| + |f_{N+1}(x) - f(x)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. □

Theorem 13.2. *Suppose that f_n is a sequence of real valued functions which are integrable on $[a, b]$, and that (f_n) converges uniformly to a function f on $[a, b]$. Then f is integrable on $[a, b]$ and $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.*

Proof. Assume that f is integrable on $[a, b]$. Given $\varepsilon > 0$, let N be such that, if $n > N$ then $|f_n(x) - f(x)| < \varepsilon/2(b - a)$ for all x in $[a, b]$. Therefore, if $n > N$, then $-\varepsilon < \int_a^b f_n - \int_a^b f < \varepsilon$, confirming that $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.

If $|f(x) - g(x)| < \varepsilon$ for all x in an interval $[t, t']$, then $g(x) - \varepsilon \leq f(x) \leq g(x) + \varepsilon$ and thus l. u. b. $g - \varepsilon \leq$ l. u. b. $f \leq$ l. u. b. $g + \varepsilon$ on $[t, t']$.

Therefore, if P is a partition of $[a, b]$, then $|U(f_n, P) - U(f, P)| \leq \varepsilon$, and similarly, $|L(f_n, P) - L(f, P)| \leq \varepsilon$. It follows that f is integrable if all the f_n are integrable. □

Theorem 13.3. *Suppose that f_n is a sequence of functions which are differentiable on $[a, b]$, which converges (pointwise) to f. Suppose that the derivatives f_n' are integrable, and that the sequence f_n' converges uniformly on $[a, b]$ to some continuous function g. Then f is differentiable, and $f'(x) = \lim_{n\to\infty} f_n'(x)$.*

Proof. Apply Theorem 13.2 and Theorem 11.2 on $[a, x]$ and obtain

$$\begin{aligned} \int_a^x g &= \lim_n \int_a^x f_n' \\ &= \lim_n [f_n(x) - f_n(a)] \\ &= f(x) - f(a). \end{aligned}$$

Since g is continuous, it follows from Theorem 11.2 that $f(x) = \int_a^x g - f(a)$ is differentiable, and that $f'(x) = g(x)$ for all x in $[a, b]$. □

Corollary 1. *In Theorem 13.3, the sequence (f_n) converges uniformly to f.*

Proof. The proof of Theorem 13.3 shows that $g = f'$, and its hypothesis state that $f_n' \to f' = g$ uniformly. Since $f_n(x) = f(a) + \int_a^x f_n'$ and $f(x) = f(a) + \int_a^x f'$, if n is such that $|f_n'(x) - f'(x)| \leq \varepsilon/2$ for all x in $[a, b]$, then

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(a) - f(a)| + \int_a^n |f_n' - f| \\ &\leq \frac{\varepsilon}{2} + \int_a^x \frac{\varepsilon}{2(b-a)} \\ &< \varepsilon \end{aligned}$$

for any x in $[a, b]$. □