Math 350. Study Guide

The final exam questions will be similar in style to those questions in quizzes and in homework. The outline below, together with textbook, class notes, homeworks and answer sheets, should help you organize your study for this test. The questions included should help you test your understanding of the material. They may not be exactly the questions in the final exam.

A. Topology

Sequences and limits. Algebra of limits. Open and closed sets. Behavior of open and closed under union and intersection. Cluster points. Least upper bound and greatest lower bound of sets of real numbers. Continuous functions.

(A.1) Find the least upper bound and the greatest lower bound, if they exist, of the set $\{x \in \mathbf{R} \mid x < 0 \text{ and } x^2 + x - 1 < 0\}$. Does this set have a maximum? a minimum?

(A.2) If $A \neq \emptyset$ is bounded below, let B be the set of all lower bounds of A. Then $B \neq \emptyset$, B is bounded above, and $\text{l. u. b.}\, B = \text{g. l. b.}\, A$.

(A.3) Let (a_n) be a bounded sequence of real numbers. Prove that if p is the only accumulation point of the set $A = \{a_n \mid n \text{ in } \mathbf{N}\}$, then the sequence (a_n) converges and $\lim_{n\to\infty} a_n = p$. Is this property true for unbounded sequences? (Prove or give a counterexample.)

(A.4) Let $f : \mathbf{R} \to \mathbf{R}$ be the function given by $f(x) = x$ if x is rational, and $f(x) = -x$ if x is irrational. Prove that f is continuous at $p = 0$ and discontinuous at any $p \neq 0$.

(A.5) Let $f : \mathbf{R} \to \mathbf{R}$ be continuous, and let a be any number. Prove that the set $\{x \mid f(x) = a\}$ is closed and that the set $\{x \mid f(x) > a\}$ is open.

B. Compact Sets

Compact sets. The Bolzano-Weierstrass Theorem: compact if and only if closed and bounded. Preservation (or not) of compactness under unions and intersections. Lower and upper bounds, supremum and infimum, and maximum and minimum of sets of real numbers.

Continuous functions take compact sets to compact sets. A continuous function on an interval $[a, b]$ attains it maximum and minimum values.

(B.1) Prove or give a counterexample: (a) The union of compact sets is compact. (b) The intersection of compact sets is compact.

(B.2) Give an example of a function on $[0, 1]$ which is bounded but attains neither maximum nor minimum.

(B.3) Prove or give a counterexample: if f is continuous and K is compact, then $f^{-1}K$ is compact.

(B.4) Suppose that $f : \mathbf{R} \to \mathbf{R}$ is continuous with $f(x) > 0$ for all x, and $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$. Prove that f attains a maximum.

(B.5) Suppose that f is continuous on $[a, b]$ and let x be any number. Prove that there is a point on the graph of f which is closest to the point $(x, 0)$ on the plane; that is, prove that there is y in $[a, b]$ such that the distance from $(x, 0)$ to $(y, f(y))$ is $\leq$ distance from $(x, 0)$ to $(z, f(z))$ for any z in $[a, b]$. Prove also that this assertion is false if $[a, b]$ is replaced by (a, b), but is true if $[a, b]$ is replaced by $\mathbf{R}$.

C. Uniform Continuity

Uniform continuity: what it is and how it is different from continuity. Uniform continuity and compactness. Uniform continuity and Cauchy sequences.

(C.1) Let $f(x) = x^a$. For which of the following values of a is f uniformly continuous on $[0, \infty)$: $a = 1/3, 1/2, 1, 2, 3$?

(C.2) Find a function which is continuous and bounded on $[0, \infty)$, but not uniformly continuous on $[0, \infty)$.

(C.3) Prove or give a counterexample: if f is continuous and bounded on $(0, 1]$, then f is uniformly continuous on $(0, 1]$.

(C.4) Suppose that f is uniformly continuous on A, g is uniformly continuous on B, and $f(x)$ is in B for all x in A. Prove that $g \circ f$ is uniformly continuous on A.

(C.5) Prove or give a counterexample: (i) If f is uniformly continuous, then f takes Cauchy sequences to Cauchy sequences; (ii) If f takes Cauchy sequences to Cauchy sequences, then f is uniformly continuous.

D. Connected sets. Intermediate Value Theorem

Connected sets. Intervals are connected. Intermediate Value Theorem: statement, proof, applications. Continuous functions take intervals to intervals.

(D.1) Prove or give a counterexample: (i) if $f : \mathbf{R} \to \mathbf{R}$ is continuous and I is an interval, then $f(I)$ is also an interval; (ii) if $f : \mathbf{R} \to \mathbf{R}$ is continuous and J is an interval, then $f^{-1}(J)$ is an interval.

(D.2) A set A of real numbers is called *dense* if very nonempty open interval contains a point of A. (For example, the set of all rational numbers is dense in $\mathbf{R}$.) Prove that if $f : \mathbf{R} \to \mathbf{R}$ is continuous and $f(x) = 0$ for all x in a dense set A, then $f(x) = 0$ for all x.

(D.3) Prove that if f is continuous on an interval I and $f(x)$ is rational for all x in I, then f is constant on I.

(D.4) How many continuous functions $f : \mathbf{R} \to \mathbf{R}$ are there which satisfy $(f(x))^2 = x^2$ for all x?

(D.5) Let f be continuous on $[a, b]$ and $f(a) < 0 < f(b)$. We proved in class that there is a smallest x in $[a, b]$ with $f(x) = 0$. Is there necessarily a second smallest x in $[a, b]$ with $f(x) = 0$? Prove that there is a largest x in $[a, b]$ with $f(x) = 0$.

E. The Derivative and its Significance

Definition of derivative. Calculations of derivatives, including the chain rule. Differentiability and continuity. The derivative and maxima and minima. Critical points of functions. Rolle's theorem. Mean Value Theorem. Applications: estimates of functions, constant functions, increasing and decreasing.

(E.1) If $f(x) = |x|^3$, find $f'(x)$ and $f''(x)$ (to be sure, f'' is the derivative of f'). Does $f'''(x)$ exist for all x?

(E.2) Prove that: (i) if f has a critical point at a and $f''(a) > 0$, then f has a local minimum at a; (ii) if f has a local minimum at a and $f''(a)$ exists, then $f''(a) \geq 0$.

(E.3) Prove that if f is differentiable on (a, b) and f' is bounded on (a, b), then f is uniformly continuous on (a, b).

(E.4) Prove that if f is continuous at a and $|f|$ is differentiable at a, then f is also differentiable at a. How are $f'(a)$ and $|f|'(a)$ related?

(E.5) Let $a \neq 0$ and let x be such that $(x + a)^n = x^n + a^n$. Prove that if n is even, then $x = 0$, and that if n is odd, then $x = 0$ or $x = -a$.

F. The Integral. Fundamental Theorem of Calculus

Partitions of closed intervals. Upper and lower sums of a function for a partition. Definition of Integrable functions. Continuous implies integrable. Linearity of the integral. Comparison of integrals. The Fundamental Theorem of Calculus.

(F.1) Let $f(x) = 1$ if $x \neq 0$ and $f(0) = 0$. Determine if f is integrable on $[-1, 1]$ and find $\int_{-1}^{1} f$ if possible.

(F.2) If $f \geq 0$ is continuous on $[a, b]$ and $f(x) \neq 0$ for some x in $[a, b]$, then $\int_a^b f > 0$.

(F.3) Prove that

$$\frac{1}{7\sqrt{2}} \leq \int_0^1 \frac{x^6}{\sqrt{1 + x^2}}\, dx \leq \frac{1}{7}.$$

(F.4) Assume that f is continuous. Find $F'(x)$ if $F(x) = \int_0^x x f(t)\, dt$.

(F.5) Prove that if f is increasing on $[1, \infty)$ then

$$f(1) + f(2) + \cdot + f(n-1) \leq \int_1^n f \leq f(2) + f(3) + \cdot + f(n)$$

for every integer $n \geq 1$ such that f is integrable on $[1, n]$.

G. The Exponential and the Logarithm

Definition of log and exp, domain, range, increasing. Continuity and derivatives. Estimates of growth.

(G.1) Let $f(x) = \log|x|$ for $x \neq 0$. Prove that $f'(x) = 1/x$ for $x \neq 0$.

(G.2) Prove that $\log x < x - 1$ for all $x > 0$, except $x = 1$.

(G.3) Let e be the number such that $\log e = 1$. Prove that $\frac{5}{2} < e < 3$.

(G.4) Prove that if f is differentiable and $f'(x) = f(x)$ for all real numbers x, then there is a number c such that $f(x) = c \cdot \exp(x)$ for all x.

(G.5) Prove that $\lim_{x \to \infty} \frac{x^n}{\exp(x)} = 0$ for any $n > 0$.

H. Sequences of Functions

Sequences of functions. Pointwise convergence. Uniform convergence. Interchange of limit operations.

(H.1) Determine if the following sequences of functions converge pointwise or uniformly on the given interval. In case that there is pointwise convergence, then you must also identify the limit function.

(i) $f_n(x) = \frac{\sin nx}{n}$, $0 \leq x \leq 1$.

(ii) $g_n(x) = \frac{1}{n} \exp(-nx)$, $0 \leq x < \infty$.

(iii) $h_n(x) = nx(1 - x^2)^n$, $0 \leq x \leq 1$.

(H.2) Suppose that $f_1, f_2, f_3, \cdots$ is a sequence of continuous functions on $[0, 1]$ that converges uniformly to f. Prove that $\lim_{n \to \infty} \int_{1/n}^{1} f_n = \int_0^1 f$. Is this true if the convergence is not uniform?

(H.3) Suppose that $f_1, f_2, f_3, \cdots$ is a sequence of continuous functions on $[a, b]$ that converges uniformly to f. Prove that if $x_n \to x$ in $[a, b]$, then $\lim_{n \to \infty} f_n(x_n) = f(x)$. Is this true without assuming that all f_n are continuous.

(H.4) Find a uniformly convergent sequence of differentiable functions $f_n : (0, 1) \to \mathbf{R}$ such that the sequence $f_1', f_2', f_3', \cdots$ does not converge.

(H.5) Prove that the sequence of functions E_n given by $E_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ converges uniformly to the exponential function exp on the closed interval $[0, 1]$.