Math 350. Quiz 8. Date: 4/29/09

Problem 1. Let $f(x) = 0$ if $x = 0$ and $f(x) = 1$ if $x \neq 0$. Determine if f is integrable on $[-1, 1]$, and evaluate $\int_{-1}^{1} f$ if possible.

Solution. For any positive integer n, let P_n be the partition of $[-1, 1]$ given by $P_n = \{-1, -1/n, 1/n, 1\}$. Then

$$\begin{aligned} L(f, P_n) &= 1 \cdot \left(-\frac{1}{n} - (-1)\right) + 0 \cdot \left(\frac{1}{n} - \left(-\frac{1}{n}\right)\right) + 1 \cdot \left(1 - \frac{1}{n}\right) \\ &= 2 - \frac{2}{n} \end{aligned}$$

and

$$\begin{aligned} U(f, P_n) &= 1 \cdot \left(-\frac{1}{n} - (-1)\right) + 1 \cdot \left(\frac{1}{n} - \left(-\frac{1}{n}\right)\right) + 1 \cdot \left(1 - \frac{1}{n}\right) \\ &= 2 \end{aligned}$$

Therefore, given $\epsilon > 0$, let n be a positive integer such that $\frac{2}{n} < \epsilon$. Then

$$U(f, P_n) - L(f, P_n) = \frac{2}{n} < \epsilon,$$

proving that f is integrable on $[-1, 1]$.

To evaluate $\int_{-1}^{1} f$, note that for any positive integer n,

$$2 - \frac{2}{n} = L(f, P_n) \leq \int_{-1}^{1} f \leq U(f, P_n) = 2,$$

which implies that

$$\int_{-1}^{1} f = 2.$$

□