4 Limits

Note 4.1. Let E denote either the real numbers $\mathbf{R}$ or the euclidean space $\mathbf{R}^k$. For $x = (x_1, x_2, \cdots, x_k)$ and $y = (y_1, y_2, \cdots, y_k)$ in $\mathbf{R}^k$, $|x - y|$ denotes the Euclidean distance between x and y:

$$|x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_k - y_k)^2}.$$

In particular, if $k = 1$, then $\mathbf{R}^k = \mathbf{R}$ is the set of real numbers, and $|x - y|$ is just the absolute value of the difference $x - y$.

Definition 4.1. Let f be a function whose domain $\operatorname{dom}(f)$ is a subset of E, and let p be a cluster point of $\operatorname{dom}(f)$. We say that the limit of f when x approaches p is a, and write $\lim_{x\to p} f(x) = a$, if for any $\varepsilon > 0$ there is $\delta > 0$ such that if x is in the domain of f and $0 < |x - p| < \delta$, then $|f(x) - a| < \varepsilon$.

Note 4.2. To say that f does not have limit at p means:

> "For every a, there exists $\varepsilon > 0$ such that for every $\delta > 0$ there is some x which satisfies $0 < |x - p| < \delta$ but not $|f(x) - a| < \varepsilon$."

Theorem 4.1. *A function has at most one limit at any cluster point of its domain.*

Theorem 4.2. *A function f has the limit a at p if and only if for every sequence $p_n \to p$, with $p_n \neq p$, the sequence $f(p_n)$ converges to a.*

Proof. Suppose that $\lim_{x\to p} f(x) = a$. Let $p_1, p_2, \cdots$ be a sequence in $\operatorname{dom} f$ such that $p_n \to p$. Given $\varepsilon > 0$, let $\delta > 0$ be such that if $0 < |x - p| < \delta$ then $|f(x) - a| < \varepsilon$. Because $p_n \to p$, for this δ, there is N such that if $n > N$ then $|p_n - p| < \delta$. Therefore, if $n > N$, $|f(p_n) - a| < \varepsilon$.

Suppose that f does not approach a near p. Then there is $\varepsilon > 0$ such that for any n, there is p_n in $\operatorname{dom} f$ such that $0 < |p_n - p| < 1/n$ and that $|f(p_n) - a| \geq \varepsilon$. The sequence p_n converges to p, has $p_n \neq p$, but $f(p_n)$ does not converge to a. $\square$

Theorem 4.3. *If $\lim_{x\to p} f(x) \neq a$, then there is a $\delta > 0$ such that $f(q) \neq a$ for every $q \neq p$ in $B(p, \delta) \cap \operatorname{dom}(f)$.*

Proof. Let $\lim_{x\to p} f(x) = l$ and let $\varepsilon = |l - a| > 0$. There there is $\delta > 0$ such that $f(B(p, \delta) \cap \operatorname{dom}(f) \setminus \{p\}) \subset B(l, \varepsilon)$. Therefore, $|f(q) - a| \geq \varepsilon > 0$ for any $q \neq p$ in $B(p, \varepsilon) \cap \operatorname{dom}(f)$. $\square$

Theorem 4.4. *Suppose that f and g are functions and that p is a cluster point of* $\operatorname{dom} f \cap \operatorname{dom} g$. *If $\lim_{x\to p} f(x) = a$ and $\lim_{x\to p} g(x) = b$, then*

(a) $\lim_{x\to p} (f + g)(x) = a + b$.

(b) $\lim_{x\to p} (f \cdot g)(x) = a \cdot b$.

(c) *Moreover, if g is real valued and $b \neq 0$, then $\lim_{x\to p} (1/g)(x) = 1/b$.*

Proof. The technique of the proof is the same as for sequences. One technical point is that the functions f and g need not have the same domain, so $f \pm g$ and $f \cdot g$ are defined in the intersection of the domains of f and g. If p is a cluster point of $\operatorname{dom} f \cap \operatorname{dom} g$, then p is a cluster point of $\operatorname{dom} f$ and of $\operatorname{dom} g$ (but not conversely). $\square$