11 Fundamental Theorem of Calculus

Theorem 11.1. *Suppose that f is integrable on $[a,b]$, and define F on $[a,b]$ by $F(x) = \int_a^x f$. Then F is continuous on $[a,b]$.*

Proof. Because f is integrable on $[a,x]$ for all x in $[a,b]$, $F(x)$ is defined for all x in $[a,b]$.

Because f is integrable on $[a,b]$, it is bounded on $[a,b]$, so there is M such that $-M \leq f(x) \leq M$ for all x in $[a,b]$. Let c be a number in $[a,b]$ and let $h \neq 0$ be any number such that $a \leq c+h \leq b$. Then $F(c+h)-F(c) = \int_{c+h}^x f - \int_c^h f = \int_c^{c+h} f$, and so, by Theorem 10.6 (c), $-M|h| \leq F(c+h)-F(c) \leq M|h|$, which implies that $\lim_{h\to 0} F(c+h) - f(c) = 0$. □

Theorem 11.2. *If f is integrable on $[a,b]$ and continuous at c ($a < c < b$), then $F(x) = \int_a^x f$ is differentiable at c and $F'(c) = f(c)$.*

Proof. We have to show that $\lim_{h\to 0} \frac{F(c+h)-F(c)}{h} = f(c)$. For any $h \neq 0$ such that $a \leq c+h \leq b$, let m_h be the g. l. b. $f(x)$ for x between c and $c+h$, and let M_h be the l. u. b. $f(x)$ for x between c and $c+h$ Then $m_h \cdot h \leq F(c+h)-F(c) \leq M_h \cdot h$ if $h > 0$ and $m_h \cdot (-h) \leq F(c+h) - F(c) \leq M_h \cdot (-h)$ if $h < 0$. In either case it obtains that

$$m_h \leq \frac{F(c+h)-F(c)}{h} \leq M_h,$$

and because f is continuous at c, $\lim_{h\to 0} m_h = \lim_{h\to 0} M_h = f(c)$, proving that $F'(c) = f(c)$ as advertised. □

Corollary 1. *If $G(x) = \int_x^b f$, then $G(x) = \int_a^b f - \int_a^x f$, so $G'(c) = -f(c)$.*

Corollary 2. *If f is continuous on $[a,b]$ and $f = g'$ for some function g, then $\int_a^b f = g(b) - g(a)$.*

Proof. Let $F(x) = \int_a^x f$. Then F is differentiable and $F' = f = g'$ on $[a,b]$. Therefore there is a number c such that $F(x) = g(x) + c$. Evaluation at a shows that $F(x) = g(x) - g(a)$, and thus $\int_a^b f = F(b) = g(b) - g(a)$. □

Theorem 11.3. *If f is integrable on $[a,b]$ and $f = g'$, then $\int_a^b f = g(b) - g(a)$.*

Proof. Let $P = \{t_0, \cdots, t_n\}$ be a partition of $[a,b]$. By the Mean value theorem there is a point x_i in $[t_{i-1}, t_i]$ such that $g(t_i) - g(t_{i-1}) = g'(x_i)(t_i - t_{i-1})$, or $g(t_i) - g(t_{i-1}) = f(x_i)(t_i - t_{i-1})$. Thus $m_i(t_i - t_{i-1}) \leq f(x_i)(t_i - t_{i-1}) \leq M_i(t_i - t_{i-1})$, and adding these inequalities for $i = 1, \cdots, n$, we obtain $L(f,P) \leq g(b) - g(a) \leq U(f,P)$. □

Theorem 11.4 (Fundamental Theorem of Calculus)**.** *Suppose that f is a real valued continuous function on an open interval U, and let a in U. Let F be defined on U by $F(x) = \int_a^x f$ for all x in U. Then F is differentiable and $F' = f$.*

Proof. Since f is continuous on U and U is an interval, $F(x) = \int_a^x f$ is defined for all x in U. Because U is an open interval, for any $x \neq a$ in U, there is a number b in U such that x is between a and b. By Theorem 11.1, F is differentiable at x and $F'(x) = f(x)$. To show that F is differentiable at a, let $b \neq a$ be any number in U and let $G(x) = \int_b^x$. Then G is differentiable at a and $G'(a) = f(a)$. Since $F(x) = G(x) - \int_a^b f$, F is differentiable at a and $F'(a) = G'(a) = f(a)$ □