10 The Integral

Definition 10.1. A **partition** of the interval $[a,b]$ is a finite collection of points in $[a,b]$, one of which is a, one of which is b. The points in a partition can always be numbered $t_0, t_1, \cdots, t_n$, so that $a = t_0 < t_1 < \cdots < t_n = b$, and write the partition $P = \{t_0, \cdots, t_n\}$.

Definition 10.2. Let $f : [a,b] \to \mathbf{R}$ be bounded, and let $P = \{t_0, \cdots, t_n\}$ be a partition of $[a,b]$. Let m_i and M_i be defined by

$$m_i = \text{g. l. b.}\{f(x) \mid t_{i-1} \leq x \leq t_i\} \quad \text{and} \quad M_i = \text{l. u. b.}\{f(x) \mid t_{i-1} \leq x \leq t_i\}.$$

The **lower sum** of f for P, denoted by $L(f,P)$, and **upper sum** of f for P, denoted by $U(f,P)$, are defined by

$$L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \quad \text{and} \quad U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

Theorem 10.1. *Let $f : [a,b] \to \mathbf{R}$ be bounded.*

(a) *$L(f,P) \leq U(f,P)$ for any partition P.*

(b) *If the partition Q contains P, then $L(f,P) \leq L(f,Q)$ and $U(f,P) \geq U(f,Q)$.*

(c) *$L(f,P_1) \leq U(f,P_2)$ for any partitions P_1 and P_2.*

Proof. (a) For any partition P, $m_i \leq M_i$.

(b) The proof reduces to the case in which Q is obtained by adding a single point to P, and so to the case in which $P = \{a,b\}$ and $Q = \{a,t_1,b\}$.

(c) If P is a partition that contains both P_1 and P_2, then by (a) and (b), $L(f,P_1) \leq L(f,P) \leq U(f,P) \leq U(f,P_2)$. □

Definition 10.3. A real valued function f on an interval $[a,b]$ is said to be integrable on $[a,b]$ if f is bounded on $[a,b]$ and $\text{l. u. b.}\{L(f,P) \mid P \text{ partition of } [a,b]\} = \text{g. l. b.}\{U(f,P) \mid P \text{ partition of } [a,b]\}$. In this case, this common number is called the **integral** of f on $[a,b]$ and is denoted by $\int_a^b f$.

Theorem 10.2. *Let f be bounded on $[a,b]$. Then f is integrable on $[a,b]$ if and only if for every $\varepsilon > 0$ there is a partition P such that $U(f,P) - L(f,P) < \varepsilon$.*

Proof. If f is integrable, for every $\varepsilon > 0$ there are partitions P' and P'' such that $U(f,P'') - L(f,P') < \varepsilon$. If P contains both P' and P'', then $L(f,P') \leq L(f,P)$ and $U(f,P) \leq U(f,P'')$.

Conversely, from the hypothesis it follows that for every $\varepsilon > 0$, $\text{g. l. b. } U(f,P) - \text{l. u. b. } L(f,P) < \varepsilon$. □

Theorem 10.3. *If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.*

Proof. Because f is continuous and $[a,b]$ is compact, f is bounded and is also uniformly continuous. Because of this, there is $\delta > 0$ such that if $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon/2(b-a)$. Let $P = \{t_0, t_1, \cdots, t_n\}$ be any partition such that $|t_i - t_{i-1}| < \delta$. Because f is continuous and each $[t_{i-1}, t_i]$ is compact, there are x_i and y_i in $[t_{i-1}, t_i]$ such that $M_i = f(x_i)$ and $m_i = f(y_i)$, hence that $M_i - m_i < \varepsilon/(b-a)$. Therefore $U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) < \varepsilon$. □

Theorem 10.4. *Let $a < c < b$. If f is integrable on $[a,b]$, then f is integrable on $[a,c]$ and on $[c,b]$. Conversely, if f is integrable on $[a,c]$ and on $[c,b]$, then f is integrable on $[a,b]$. Finally, if f is integrable on $[a,b]$, then $\int_a^b f = \int_a^c f + \int_c^b f$.*

Proof. Suppose that f is integrable on $[a, b]$. Let P be a partition of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. we can always assume that c is in P, so P determines partitions P' of $[a, c]$ and P'' of $[c, b]$. Moreover, $U(f, P) = U(f, P') + U(f, P'')$ and similarly for $L(f, P)$. Therefore, $[U(f, P') - L(f, P')] + [U(f, P'') - L(f, P'')] < \varepsilon$, and so both terms in brackets are $< \varepsilon$ because both are ≥ 0.

Note also that $L(f, P') \leq \int_a^c f \leq U(f, P')$ and that $L(f, P'') \leq \int_a^c f \leq U(f, P'')$, so that $L(f, P) \leq \int_a^c f + \int_c^b f \leq U(f, P)$. Since this holds for any partition P (containing c), we have $\int_a^b f = \int_a^c f + \int_c^b f$.

Suppose that f is integrable on $[a, c]$ and on $[c, b]$. If $\varepsilon > 0$, then there is a partition P' of $[a, c]$ and a partition P'' of $[c, b]$ such that

$$U(f, P') - L(f, P') < \varepsilon/2 \qquad U(f, P'') - L(f, P'') < \varepsilon/2.$$

If $P = P' \cup P''$, then $U(f, P) - L(f, P) = U(f, P') + U(f, P'') - L(f, P') - L(f, P'') < \varepsilon$. □

Definition 10.4. If f is integrable on $[a, b]$, we write $\int_b^a f$ to mean $-\int_a^b f$. If $a = b$, write $\int_a^a f = 0$. Theorem 10.4 can be restated thus: if a, b, c are any three real numbers and any two of $\int_a^b f$, $\int_b^c f$, $\int_a^c f$ exist, so does the third, and $\int_a^b f + \int_b^c f + \int_c^a = 0$.

Theorem 10.5. *Let f and g be integrable on $[a, b]$. Then*

(a) *If c, d are constants, then $cf + dg$ is integrable and* $\int_a^b (cf + dg) = c \int_a^b f + d \int_a^b g$.

(b) *$|f|$ is integrable on $[a, b]$ and* $\left| \int_a^b f \right| \leq \int_a^b |f|$.

(c) *$f \cdot g$ is integrable.*

Proof. (a) If P is any partition of $[a, b]$, then then $L(cf, P) = cL(f, P)$ and $U(cf, P) = cU(f, P)$ if $c \geq 0$, and $L(cf, P) = cU(f, P)$ and $U(cf, P) = cL(f, P)$ if $c \leq 0$. Therefore, cf is integrable and $\int_a^b (cf) = c \int_a^b f$.

For any partition P, $m_i(f) + m_i(g) \leq m_i(f + g) \leq M_i(f + g) \leq M_i(f) + M_i(g)$, and so $U(f + g, P) - L(f + g, P) \leq U(f, P) - L(f, P) + U(g, P) - L(g, P)$. If f and g are integrable on $[a, b]$, there a re partitions P and P' such that $U(f, P') - L(f, P') < \varepsilon/2$ and $U(g, P'') - Lg, P'') < \varepsilon/2$. If $P = P' \cup P''$, then $U(f, P) + U(g, P) - [L(f, P) + L(g, P)] < \varepsilon$, and consequently, $U(f + g, P) - L(f + g, P) < \varepsilon$, proving that $f + g$ is integrable. Furthermore, $L(f, P) + L(g, P) \leq L(f + g, P) \leq \int_a^b (f + g) \leq U(f + g, P) \leq U(f, P) + U(g, P)$ and also $L(f, P) + L(g, P) \leq \int_a^b f + \int_a^b g \leq U(f, P) + g(U, P)$. Since this is true for all partitons P, it follows that $\int_a^b (f + g) = \int_a^b f + \int_a^b g$. □

Theorem 10.6. (a) *If $f \geq 0$ on $[a, b]$, then* $\int_a^b f \geq 0$.

(b) *If $f \leq g$ on $[a, b]$, then* $\int_a^b f \leq \int_a^b f$.

(c) *If $m \leq f \leq M$ on $[a, b]$, then* $m(b - a) \leq \int f \leq M(b - a)$

Proof. (a) If P is any partition of $[a, b]$, then $L(f, P) \geq 0$.

(b) Apply (a) to $g - f$.

(c) If P is a partition of $[a, b]$, then $m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a)$.

□