Problem 1. (a) Prove that if f and g are continuous on $[a,b]$, if $m \le f(x) \le M$ for all x in $[a,b]$, and if g is nonnegative on $[a,b]$, then

$$m\int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le M\int_a^b g(x)\,dx.$$

(b) Use Part (a) to prove that

$$\frac{1}{7\sqrt{2}} \le \int_0^1 \frac{x^6}{\sqrt{1+x^2}}\,dx \le \frac{1}{7}.$$

Problem 2. Prove that

$$\frac{3}{8} \le \int_0^{1/2} \sqrt{\frac{1-x}{1+x}}\,dx \le \frac{\sqrt{3}}{4}.$$

Solution. Note that

$$\sqrt{\frac{1-x}{1+x}} = \frac{1-x}{\sqrt{1-x^2}}.$$

Therefore,

$$\begin{aligned}\left[\min \text{ of } \frac{1}{\sqrt{1-x^2}} \text{ on } [0,1/2]\right]\int_0^{1/2}(1-x)\,dx &\le \int_0^{1/2}\sqrt{\frac{1-x}{1+x}}\,dx \\ &\le \left[\max \text{ of } \frac{1}{\sqrt{1-x^2}} \text{ on } [0,1/2]\right]\int_0^{1/2}(1-x)\,dx\end{aligned}$$

Because the function $g(x) = \dfrac{1}{\sqrt{1-x^2}}$ is decreasing on $[0,1/2]$ with $g(0)=1$ and $g(1/2) = 2/\sqrt{3}$, we obtain

$$\int_0^{1/2}(1-x)\,dx \le \int_0^{1/2}\sqrt{\frac{1-x}{1+x}}\,dx \le \frac{2}{\sqrt{3}}\int_0^{1/2}(1-x)\,dx.$$

Now

$$\int_0^{1/2}(1-x)\,dx = x - \frac{x^2}{2}\Big|_0^{1/2} = \frac{3}{8}$$

and thus

$$\frac{3}{8} \le \int_0^{1/2} \sqrt{\frac{1-x}{1+x}}\,dx \le \frac{\sqrt{3}}{4}.$$

□

Problem 3. Let f be integrable on $[a,b]$, let c be in (a,b), and let

$$F(x) = \int_a^x f, \quad a \le x \le b.$$

For each of the following statements, give either a proof or a counterexample.

(a) If f is differentiable at c, then F is differentiable at c.

(b) If f is differentiable at c, then F' is continuous at c.

(c) If f' is continuous at c, then F' is continuous at c.

Problem 4. If $f(x) = \displaystyle\int_0^x \sqrt{t+t^6}\cdot dt$, find $f'(3)$.

Problem 5. Let $f(x) = \int_1^x (1 + \sin(\sin t))\,dt$. Compute $f'(x)$ and prove that f is increasing.

Solution. By the Fundamental Theorem of Calculus, $f'(x) = 1 + \sin(\sin(x))$. Since $-1 \leq \sin(x) \leq 1$ for all x, $f'(x) \geq 0$, so f is increasing. In fact, f is strictly increasing because $f'(x) > 0$. Indeed, $-1 < \sin(t) < 1$ for $-1 < t < 1$, so $-1 < \sin(\sin(x)) < 1$. □

Problem 6. Find the derivatives of the following functions.

(a) $F(x) = \int_x^b \frac{1}{1 + t^2 + \sin^2 t}\,dt$.

(b) $F(x) = \int_a^b \frac{x}{1 + t^2 + \sin^2 t}\,dt$.

Problem 7. Prove that

$$\int_0^x \frac{1}{1+t^2}\,dt = c + \int_{1/x}^0 \frac{1}{1+t^2}\,dt$$

for some constant c.

Solution. The functions $\int_0^x \frac{1}{1+t^2}\,dt$ and $\int_{1/x}^0 \frac{1}{1+t^2}\,dt$ have the same derivative, therefore, they differ by a constant. □

Problem 8. Prove that if h is continuous, f and g are differentiable, and

$$F(x) = \int_{f(x)}^{g(x)} h(t)\,dt,$$

then $F'(x) = h(g(x)) \cdot g'(x) - h(f(x)) \cdot f'(x)$.

Problem 9. Find all the continuous functions f satisfying

$$\int_0^x f = (f(x))^2 + C,$$

for some constant C.

Solution. Because f is continuous, by the fundamental theorem of calculus, $\int_0^x f$ is differentiable everywhere (its derivative at x is $f(x)$. Therefore f^2 is differentiable everywhere, and its derivative at x is $f(x)$. It follows that if $f(x) \neq 0$, then f is differentiable at x if $f(x) \neq 0$, and $f(x) = 2f(x)f'(x)$.

Therefore, on each interval where f is everywhere > 0 or everywhere < 0, f is of the form $f(x) = x/2 + D$, for some constant D (which depends on the interval).

We analyze the set of points x where $f(x) = 0$, eventually proving that this set is a closed interval (or a point). First note that there is some x such that $f(x) = 0$. Indeed, if $f(x) \neq 0$ for every x, then $f(x) = x/2 + D$ for some constant D, but then $f(-2D) = 0$.

Suppose that there are two distinct numbers $a < b$ such that $f(a) = f(b) = 0$, and suppose that $f(x_0) > 0$ for some x_0 in $[a, b]$. Let $\alpha = \text{l.u.b.}\{x \mid a \leq x \leq x_0; f(x) = 0\}$ and $\beta = \text{g.l.b.}\{x \mid x_0 \leq x \leq b; f(x) = 0\}$. Then $f(\alpha) = f(\beta) = 0$ because f is continuous, and $\alpha \neq \beta$ because $\alpha < x_0 < \beta$. Furthermore, $f(x) > 0$ for any x in (α, β) by the Intermediate Value Theorem. But then, $f(x) = x/2 + D$ for some constant D and all x in (α, β). This contradict the continuity of f because it implies that $\alpha/2 + D = f(\alpha) = 0 = f(\beta) = \beta/2 + D$. or $\alpha = \beta$. It follows that if f vanishes at two distinct points a and b, then $f(x) = 0$ for all x in $[a, b]$.

Therefore, the set set $Z = \{x \mid f(x) = 0\}$ is a closed interval. There are several possibilities for Z:

(a) Z is a single point $Z = \{a\}$. Then $f(x) = x/2 - a/2$.

(b) $Z = [a, b]$, with $a \neq b$. Then

$$f(x) = \begin{cases} x/2 - a/2, & x < a \\ 0, & a \leq x \leq b \\ x/2 - b/2, & x > b. \end{cases}$$

(c) $Z = (-\infty, b]$. Then

$$f(x) = \begin{cases} 0, & x \le b \\ x/2 - b/2, & x > b \end{cases}$$

(d) $Z = [a, \infty)$. Then

$$f(x) = \begin{cases} x/2 - a/2, & x < a \\ 0, & a \le x. \end{cases}$$

(e) $Z = (-\infty, \infty)$. Then $f(x) = 0$ for all x.

Note that $C = -f(0)^2/4$. □

Problem 10. Prove that if f and g have continuous derivatives on $[a, b]$, then

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$$

Problem 11. Let $f(x) = \log|x|$ for $x \neq 0$. Prove that $f'(x) = 1/x$ for $x \neq 0$.

Solution. The function g on $\{x \neq 0\}$ defined by $g(x) = |x|$ is differentiable with $g'(x) = 1$ is $x > 0$ and $g'(x) = -1$ if $x < 0$. The function f is the composite $f = \log \circ g$, and it is thus differentiable. By the chain rule, $f'(x) = \log'(g(x)) \cdot g'(x)$. Therefore, $f'(x) = \frac{1}{x}$ if $x > 0$, and $f'(x) = \frac{1}{-x}(-1) = \frac{1}{x}$ if $x < 0$. □

Problem 12. Let e be the number such that $\log e = 1$. Prove that $\frac{5}{2} < e < 3$.

Problem 13. (a) Prove that $\frac{\log(x)}{x} \le \frac{1}{\sqrt{x}} \int_1^x \frac{1}{t^{3/2}} \cdot dt$, for all $x \ge 1$.

(b) Prove that $\lim_{x\to\infty} \frac{\log(x)}{x} = 0$.

(c) Prove that $\lim_{x\to\infty} \frac{\log(x)}{x^n} = 0$, for any $n > 0$.

Solution. (a) If $1 \le t \le x$, then $\frac{1}{t} = \frac{\sqrt{t}}{t^{3/2}} \le \frac{\sqrt{x}}{t^{3/2}}$, so that, by Problem 1(a),

$$\log(x) = \int_1^x \frac{1}{t}\, dt \le \sqrt{x} \int_1^x \frac{1}{t^{3/2}}\, dt.$$

□

Problem 14. Prove that if f is differentiable and $f'(x) = f(x)$ for all real numbers x, then there is a number c such that $f(x) = c \cdot \exp(x)$ for all x.

Solution. The function $g(x) = f(x)/\exp(x)$ is differentiable at all x with $g'(x) = \frac{f'(x)\exp(x) - f(x)\exp(x)}{\exp(2x)} = 0$. Therefore, $g(x) = c$ for some constant c and all x, or $f(x) = c\exp(x)$. □

Problem 15. Prove that if$f(x) = \int_0^x f(t)\, dt$, then $f(x) = 0$ for all x.

Solution. The function f is continuous and thus differentiable at all x, with derivative $f'(x) = f(x)$. By the previous problem, $f(x) = c\exp(x)$ for some constant c. Since also $f(0) = 0$, the constant $c = 0$, of $f(x) = 0$ for all x. □

Problem 16. Prove that $\lim_{x\to\infty} \frac{x^n}{\exp(x)} = 0$ for any $n > 0$.

Problem 17. Let $f(x) = \dfrac{\exp(x)}{x^n}$ for $x > 0$.

(a) Find the minimum value of $f(x)$ for $x > 0$, and conclude that $f(x) > \dfrac{\exp(n)}{n^n}$ for all $x > n$.

(b) Using the expression for $f'(x)$ found in (a), prove that $f'(x) > \dfrac{\exp(n+1)}{(n+1)^{n+1}}$ for $x > n+1$.

Problem 18. Let $f(x) = \dfrac{1}{\sqrt{1+x^2}}$ and let $F(x) = \displaystyle\int_0^x f$.

(a) Prove that F is uniformly continuous on $\mathbf{R}$.

(b) Prove that $F(-x) = -F(x)$.

(c) Prove that F is strictly increasing on $\mathbf{R}$.

(d) Prove that $F(x) \geq \log\sqrt{x}$ for all $x \geq 1$.

(e) Prove that F take on all real values: if y is any number, there is a number x such that $F(x) = y$.

Solution. (a) By the Fundamental Theorem of Calculus, $F'(x) = \dfrac{1}{1+x^2}$. Since $0 < F'(x) \leq 1$ for all x, the Mean Value Theorem implies that

$$|F(x) - F(y)| \leq |x - y|$$

for any numbers x and y.

(c) $F'(x) > 0$ for all x.

(d) We have $\dfrac{1}{2t} < \dfrac{1}{\sqrt{1+t^2}}$ for any $t \geq 1$. Therefore, for $x \geq 1$,

$$\begin{aligned} F(x) &= \int_0^x \frac{1}{1+t^2}dt \\ &= \int_0^1 \frac{1}{1+t^2}dt + \int_1^x \frac{1}{1+t^2}dt \\ &> \int_1^x \frac{1}{1+t^2}dt \\ &\geq \int_1^x \frac{1}{2t}dt \\ &= \frac{1}{2}\log(x) \\ &= \log\sqrt{x} \end{aligned}$$

(e) By (d), $\lim_{x\to\infty} F(x) = \infty$. By (d) and (a), $\lim_{x\to-\infty} F(x) = -\infty$. This implies that given any number y, let $a < b$ (F is increasing) be such that $F(a) < y-1$ and $F(b) > y+1$. By the Intermediate Value Theorem for F on $[a, b]$, there is x in (a, b) such that $F(x) = y$.

□

Problem 19. Let F be the function constructed in Problem 18. Let $S(x)$ be defined by $S(x) = y$ if and only if $F(y) = x$ (that is, S is the inverse function of F).

(a) Prove that S is differentiable.

(b) Prove that $S'(x) = \sqrt{1 + S^2(x)}$ for all numbers x.

(c) Prove that $S''(x) = S(x)$.

Problem 20. Let S be the function constructed in Problem 19 and let $C(x) = S'(x)$. Prove that $S(x) + C(x) = \exp(x)$ for all x.