Problem 1. (a) Prove that if f and g are continuous on $[a, b]$, if $m \leq f(x) \leq M$ for all x in $[a, b]$, and if g is nonnegative on $[a, b]$, then

$$m \int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M \int_a^b g(x)\,dx.$$

(b) Use Part (a) to prove that

$$\frac{1}{7\sqrt{2}} \leq \int_0^1 \frac{x^6}{\sqrt{1+x^2}}\,dx \leq \frac{1}{7}.$$

Problem 2. Prove that

$$\frac{3}{8} \leq \int_0^{1/2} \sqrt{\frac{1-x}{1+x}}\,dx \leq \frac{\sqrt{3}}{4}.$$

Problem 3. Let f be integrable on $[a, b]$, let c be in (a, b), and let

$$F(x) = \int_a^x f, \quad a \leq x \leq b.$$

For each of the following statements, give either a proof or a counterexample.

(a) If f is differentiable at c, then F is differentiable at c.

(b) If f is differentiable at c, then F' is continuous at c.

(c) If f' is continuous at c, then F' is continuous at c.

Problem 4. If $f(x) = \int_0^x \sqrt{t + t^6} \cdot dt$, find $f'(3)$.

Problem 5. Let $f(x) = \int_1^x (1 + \sin(\sin t))\,dt$. Compute $f'(x)$ and prove that f is increasing.

Problem 6. Find the derivatives of the following functions.

(a) $F(x) = \int_x^b \frac{1}{1 + t^2 + \sin^2 t}\,dt.$

(b) $F(x) = \int_a^b \frac{x}{1 + t^2 + \sin^2 t}\,dt.$

Problem 7. Prove that

$$\int_0^x \frac{1}{1+t^2}\,dt = c + \int_{1/x}^0 \frac{1}{1+t^2}\,dt$$

for some constant c.

Problem 8. Prove that if h is continuous, f and g are differentiable, and

$$F(x) = \int_{f(x)}^{g(x)} h(t)\,dt,$$

then $F'(x) = h(g(x)) \cdot g'(x) - h(f(x)) \cdot f'(x)$.

Problem 9. Find all the continuous functions f satisfying

$$\int_0^x f = (f(x))^2 + C,$$

for some constant C.

Problem 10. Prove that if f and g have continuous derivatives on $[a, b]$, then

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$$

Problem 11. Let $f(x) = \log|x|$ for $x \neq 0$. Prove that $f'(x) = 1/x$ for $x \neq 0$.

Problem 12. Let e be the number such that $\log e = 1$. Prove that $\frac{5}{2} < e < 3$.

Problem 13. (a) Prove that $\frac{\log(x)}{x} \leq \frac{1}{\sqrt{x}} \int_1^x \frac{1}{t^{3/2}} \cdot dt$, for all $x \geq 1$.

(b) Prove that $\lim_{x\to\infty} \frac{\log(x)}{x} = 0$.

(c) Prove that $\lim_{x\to\infty} \frac{\log(x)}{x^n} = 0$, for any $n > 0$.

Problem 14. Prove that if f is differentiable and $f'(x) = f(x)$ for all real numbers x, then there is a number c such that $f(x) = c \cdot \exp(x)$ for all x.

Problem 15. Prove that if $f(x) = \int_0^x f(t)\,dt$, then $f(x) = 0$ for all x.

Problem 16. Prove that $\lim_{x\to\infty} \frac{x^n}{\exp(x)} = 0$ for any $n > 0$.

Problem 17. Let $f(x) = \frac{\exp(x)}{x^n}$ for $x > 0$.

(a) Find the minimum value of $f(x)$ for $x > 0$, and conclude that $f(x) > \frac{\exp(n)}{n^n}$ for all $x > n$.

(b) Using the expression for $f'(x)$ found in (a), prove that $f'(x) > \frac{\exp(n+1)}{(n+1)^{n+1}}$ for $x > n + 1$.

Problem 18. Let $f(x) = \frac{1}{\sqrt{1 + x^2}}$ and let $F(x) = \int_0^x f$.

(a) Prove that F is uniformly continuous on $\mathbf{R}$.

(b) Prove that $F(-x) = -F(x)$.

(c) Prove that F is strictly increasing on $\mathbf{R}$.

(d) Prove that $F(x) \geq \log \sqrt{x}$ for all $x \geq 1$.

(e) Prove that F take on all real values: if y is any number, there is a number x such that $F(x) = y$.

Problem 19. Let F be the function constructed in Problem 18. Let $S(x)$ be defined by $S(x) = y$ if and only if $F(y) = x$ (that is, S is the inverse function of F).

(a) Prove that S is differentiable.

(b) Prove that $S'(x) = \sqrt{1 + S^2(x)}$ for all numbers x.

(c) Prove that $S''(x) = S(x)$.

Problem 20. Let S be the function constructed in Problem 19 and let $C(x) = S'(x)$. Prove that $S(x) + C(x) = \exp(x)$ for all x.