

Problem 1. Let f and g be integrable on $[a, b]$, and let c be a constant.

(a) Prove that cf is integrable on $[a, b]$ and $\int_a^b cf = c \int_a^b f$.

(b) Prove that $f + g$ is integrable on $[a, b]$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

Problem 2. If f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and $\left| \int_a^b f \right| \leq \int_a^b |f|$.

Problem 3. Evaluate without doing any computations

(a) $\int_{-1}^1 x^3 \sqrt{1-x^2} dx$.

(b) $\int_{-1}^1 (x^5 + 1) \sqrt{1-x^2} dx$.

Solution. (a) This integral is 0, because the part from -1 to 0 is the negative of the part from 0 to 1 .

(b) The integral $\int_{-1}^1 x^5 \sqrt{1-x^2} dx = 0$ by the same reasoning, and the integral $\int_{-1}^1 \sqrt{1-x^2} dx = \frac{\pi}{2}$ because it is the area of a unit semicircle. Therefore

$$\int_{-1}^1 (x^5 + 1) \sqrt{1-x^2} dx = \int_{-1}^1 x^5 \sqrt{1-x^2} dx + \int_{-1}^1 \sqrt{1-x^2} dx = \frac{\pi}{2}.$$

□

Problem 4. Suppose that f and g are integrable on $[a, b]$. If P is a partition of $[a, b]$, let M'_i and m'_i the appropriate sup's and inf's for f on the intervals of P , define M''_i and m''_i similarly for g , and define M_i and m_i similarly for the product fg .

Assume that $f(x) \geq 0$ and $g(x) \geq 0$ for all x in $[a, b]$.

(a) Prove that $M_i \leq M'_i M''_i$ and $m_i \geq m'_i m''_i$.

(b) Prove that

$$U(P, fg) - L(P, fg) \leq \sum_{i=1}^n (M'_i M''_i - m'_i m''_i) (t_i - t_{i-1}).$$

(c) Use the fact that f and g are bounded (so that $|f(x)| \leq M$ and $|g(x)| \leq M$, for all x in $[a, b]$), to prove that

$$U(P, fg) - L(P, fg) \leq M \{U(P, f) + U(P, g) - L(P, f) - L(P, g)\}$$

(d) Prove that fg is integrable.

(e) Remove the condition that $f(x) \geq 0$ and $g(x) \geq 0$ on $[a, b]$.

Problem 5. (a) (Schwarz Inequality) Prove that

$$\left(\sum_{i=1}^n x_i y_i \right)^2 \leq \sum_{i=1}^n x_i^2 \cdot \sum_{i=1}^n y_i^2 \quad (*)$$

for real numbers x_1, \dots, x_n and y_1, \dots, y_n . There are many proofs available; one of them starts by first establishing the identity

$$\sum_{i=1}^n x_i^2 \cdot \sum_{i=1}^n y_i^2 = \left(\sum_{i=1}^n x_i y_i \right)^2 + \sum_{i < j} (x_i y_j - x_j y_i)^2.$$

- (b) Prove that equality in (*) holds if and only if there is a real number λ such that $x_i = \lambda y_i$ for all $i = 1, \dots, n$.
- (c) (Cauchy-Schwarz inequality) Suppose that f and g are integrable on $[a, b]$. Prove that

$$\left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right)\left(\int_a^b g^2\right). \tag{**}$$

- (d) If equality holds in (**), is it necessarily true that $f = \lambda g$ for some real number λ ? What if f and g are continuous?

Problem 6. Prove that if $f(x) = x^3$, then $\int_0^b f = \frac{b^4}{4}$, by considering upper and lower sums for partitions of $[0, b]$ into n equal subintervals, using the formula $1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$ for the sum of the cubes of the first n natural numbers.

Problem 7. Decide which of the following functions are integrable on $[0, 2]$, and calculate the integral sum if you can.

- (a) $f(x) = \begin{cases} x + [x], & x \text{ rational} \\ 0, & x \text{ not rational.} \end{cases}$
- (b) f is the function shown in Figure 1 (set $f(0) = 0$).

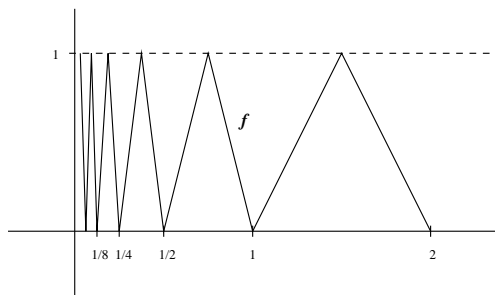


FIGURE 1

Solution. (a) This function f is not integrable because for any partition P of $[0, 2]$, the lower sum $L(f, P) = 0$ and the upper sum $U(f, P) \geq 2$.

(b) This function is integrable (you should try to write down a rigorous proof: given $\epsilon > 0$, find a partition P such that $U(f, P) - L(f, P) < \epsilon$), and the integral is the sum of the areas of the triangles:

$$\int_0^2 f = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1.$$

□

Areas If $f \geq 0$ on $[a, b]$, then $\int_a^b f$ represents the area of the region bounded by the graph of f , the horizontal axis, and the vertical lines through $(a, 0)$ and $(b, 0)$. But if f is not always ≥ 0 , considerable care must be exercised in finding such area

Similarly, if $f \geq g$ on $[a, b]$, then $\int_a^b (f - g)$ gives the area bounded by f and g from a to b , even if f and g are sometimes negative.

Problem 8. Find the areas of the regions bounded by

- (a) The graphs of $f(x) = x^2$ and $g(x) = -x^2$ and the vertical lines through $(-1, 0)$ and $(1, 0)$.
- (b) The graphs of $f = -x/2$, $g = x + 6$, $h = x^3$.

(c) The graphs of $f(x) = x^3 - 3x + 2$ and $g(x) = x + 3$.

Problem 9. (a) Prove that if f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then $\int_a^b f \geq 0$.

(b) Prove that if f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then $\int_a^b f \geq \int_a^b g$.

Solution. (a) Since $f \geq 0$, $L(f, P) \geq 0$ for any partition P . Thus $\int_a^b f \geq L(f, P) \geq 0$.

(b) It follows from part (a) because $f - g \geq 0$ and $\int_a^b f - \int_a^b g = \int_a^b (f - g)$. \square

Problem 10. (a) Give an example of an f which is integrable on $[a, b]$, satisfies $f(x) \geq 0$ for all x , and $f(x) > 0$ for some x , and yet $\int_a^b f = 0$.

(b) Suppose that $f(x) \geq 0$ for all x in $[a, b]$ and f is continuous at x_0 in $[a, b]$ and $f(x_0) > 0$. Prove that $\int_a^b f > 0$. (Hint. It suffices to find a partition P for which the lower sum $L(f, P) > 0$.)

Solution. (a) Choose x_0 in $[a, b]$ and let $f(x) = 0$ for all $x \neq x_0$ and $f(x_0) = 1$.

(b) Because f is continuous at x_0 and $f(x_0) > 0$, there is a partition P of $[a, b]$ such that $f(x) > f(x_0)/2$ for all x in some interval $[t_{i-1}, t_i]$ of P . Then

$$L(f, P) \geq (t_i - t_{i-1}) \frac{f(x_0)}{2} > 0.$$

\square

Problem 11. Suppose that f is continuous on $[a, b]$ and that $\int_a^b fg = 0$ for all continuous functions g on $[a, b]$. Prove that $f = 0$.

Solution. Take $g = f$ to obtain $\int_a^b f^2 = 0$. Since f^2 is continuous and $f^2 \geq 0$, this implies, by Problem 10(b), that $f^2(x) = 0$ for all x , or $f(x) = 0$ for all x . \square

Problem 12. Prove that

$$\int_1^a \frac{1}{x} dx + \int_1^b \frac{1}{x} dx = \int_1^{ab} \frac{1}{x} dx.$$

Problem 13. Prove that if f is continuous on $[a, b]$, then

$$\int_a^b f = (b - a)f(\xi)$$

for some number ξ in $[a, b]$; and show by example that continuity is essential.

Solution. The function $F(x) = \int_a^x f$ is continuous on $[a, b]$ and differentiable on (a, b) with $F'(x) = f(x)$. By the mean value theorem, there is ξ in (a, b) such that $F(b) - F(a) = F'(\xi)(b - a)$. \square

Limit at ∞ The symbol $\lim_{x \rightarrow \infty} f(x)$ means “the limit of $f(x)$ as x approaches ∞ .” We say that $\lim_{x \rightarrow \infty} f(x) = L$ if for every $\varepsilon > 0$ there is a number M such that, for all x ,

$$\text{if } x > M, \text{ then } |f(x) - L| < \varepsilon.$$

A similar definition applies to $\lim_{x \rightarrow -\infty} f(x) = L$.

Problem 14. The limit $\lim_{N \rightarrow \infty} \int_a^N f$, if it exists, is denoted by $\int_a^\infty f$ (or by $\int_a^\infty f(x) \cdot dx$), and called an “improper integral.”

(i) Find $\int_1^{\infty} x^r \cdot dx$ if $r < -1$.

(ii) Prove that $\int_1^{\infty} \frac{1}{x} \cdot dx$ does not exist

(iii) Does $\int_0^{\infty} \frac{1}{1+x^2} \cdot dx$ exist?

The improper integral $\int_{-\infty}^a f$ is defined as $\lim_{N \rightarrow -\infty} \int_N^a f$, as expected, but another kind of improper integral $\int_{-\infty}^{\infty} f$ is defined as $\int_0^{\infty} f + \int_{-\infty}^0 f$, provided both improper integrals exist.

(iv) Prove that $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \cdot dx$ exists.

(v) Prove that $\lim_{N \rightarrow \infty} \int_{-N}^N x \cdot dx$ exists, but the improper integral $\int_{-\infty}^{\infty} x \cdot dx$ does not exist.

(vi) Prove that the improper integral $\int_{\pi}^{\infty} \frac{\sin x}{x} \cdot dx$ exists, but $\int_{\pi}^{\infty} \frac{|\sin x|}{x} \cdot dx$ does not exist.

Solution. (i) $\int_1^N x^r dx = \frac{N^{r+1} - 1}{r+1}$. If $r < -1$, then $\lim_{N \rightarrow \infty} N^{r+1} = 0$ and so $\int_1^{\infty} x^r dx = \frac{-1}{1+r}$

(ii) The integral $\int_1^N \frac{1}{x} dx = \log N$ and $\lim_{N \rightarrow \infty} \log N$ does not exist.

(iii) Yes. Because $\frac{1}{1+x^2} > 0$ for all x , the sequence $a_N = \int_0^N \frac{1}{x^2} \cdot dx$ is increasing. Moreover, $a_N \leq 2$ for all N because

$$\int_0^N \frac{1}{1+x^2} \leq 1 + \int_1^N \frac{1}{x^2} = 2 - \frac{1}{N}$$

Therefore, the limit $\lim_{N \rightarrow \infty} \int_0^N \frac{1}{1+x^2} dx$ exists.

(iv) Use (iii) and the fact that $\int_{-N}^0 \frac{1}{1+x^2} dx = \int_0^N \frac{1}{1+x^2} dx$.

(v) $\int_{-N}^N x dx = 0$ for all N , but $\lim_{N \rightarrow \infty} \int_0^N x dx = \lim_{N \rightarrow \infty} \frac{N^2}{2}$ does not exist.

(vi) If $f(x) = \frac{-\cos(x)}{x}$, then $f'(x) = \frac{\sin(x)}{x} + \frac{\cos(x)}{x^2}$, so

$$\begin{aligned} \int_{\pi}^m \frac{\sin(x)}{x} - \int_{\pi}^m \frac{\sin(x)}{x} &= \int_{\pi}^m \frac{\sin(x)}{x} \\ &= \int_{\pi}^m f' - \int_{\pi}^m \frac{\cos(x)}{x^2} \\ &= \frac{\sin(m)}{m} - \frac{\sin(\pi)}{\pi} - \int_{\pi}^m \frac{\cos(x)}{x^2} \end{aligned}$$

Since $\left| \int_{\pi}^m \frac{\cos(x)}{x^2} \right| \leq \frac{1}{\pi} - \frac{1}{m}$, it follows that $a_n = \int_{\pi}^n \frac{\sin(x)}{x} dx$ is a Cauchy sequence, and therefore it converges, that is, the limit $\lim_{N \rightarrow \infty} \int_{\pi}^N \frac{\sin(x)}{x} dx$ exists. □

Problem 15. There is another kind of improper integral in which the interval is bounded but the function is unbounded.

- (i) If $a > 0$ and $-1 < r < 0$, find $\lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^a x^r \cdot dx$. This limit is denoted $\int_0^a x^r \cdot dx$, even though the function $f(x) = x^r$ is not bounded on $[0, a]$ (for $-1 < r < 0$), no matter how we define $f(0)$.
- (ii) Suppose that f is continuous on $[0, 1]$. Find

$$\lim_{x \rightarrow 0^+} x \int_x^1 \frac{f(t)}{t} \cdot dt.$$

- (iii) The integral $\int_0^{\infty} \frac{1}{x^2 + \sqrt{x}} \cdot dx$ does not fall into any of the two kinds of improper integrals previously described in these problems. Can you give it a meaning? (Break up the interval $(0, \infty)$ at 1.)

Solution. (i) Since $r \neq -1$, $\int_{\varepsilon}^a x^r \cdot dx = \frac{a^{r+1} - \varepsilon^{r+1}}{r+1}$, and so $\lim_{\varepsilon \rightarrow 0^+} \frac{a^{r+1} - \varepsilon^{r+1}}{r+1} = \frac{a^{r+1}}{r+1}$ because $0 < 1 + r < 1$.

(ii) Because f is continuous on $[0, 1]$ there is a constant C such that $|f(x)| \leq C$ for all x in $[0, 1]$. Thus, if $0 < x < 1$, then

$$\left| x \int_x^1 \frac{f(t)}{t} dt \right| \leq Cx \int_x^1 \frac{1}{t} dt = -Cx \log x.$$

Since $\lim_{x \rightarrow 0^+} x \log x = 0$, we obtain

$$\lim_{x \rightarrow 0^+} x \int_x^1 \frac{f(t)}{t} dt = 0.$$

□