Problem 1. (a) Prove, working directly from the definition, that if $f(x) = 1/x$, then $f'(a) = -1/a^2$, for $a \neq 0$.

(b) Prove that the tangent line to the graph of f at $(a, 1/a)$ does not intersect the graph of f, except at $(a, 1/a)$.

Problem 2. Discuss the differentiablility of $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} x^3 \sin \dfrac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Problem 3. Find f' if $f(x) = [x]$. (Here $[x]$ is the largest integer less than or equal to x.)

Problem 4. Let $f(a) = x^2$ if x is rational, and $f(x) = 0$ if x is not rational. Prove that f is differentiable at 0.

Problem 5. Let f be a function such that $|f(x)| \leq x^2$ for all x. Prove that f is differentiable at 0.

Problem 6. Prove that if f is even, then $f'(x) = -f'(-x)$. (A function f is even if $f(x) = f(-x)$.)

Problem 7. Find $f'(x)$ if

$$f(x) = \frac{1}{x - \dfrac{2}{x + \sin x}}.$$

Problem 8. Find $f'(x)$ for $-1 < x < 1$, if $f(x) = \sqrt{1 - x^2}$.

Problem 9. Suppose that a and b are two consecutive roots of a polynomial function f, but that a and b are not double roots. so that we can write $f(x) = (x - a)(x - b)g(x)$ where $g(a) \neq 0$ and $g(b) \neq 0$.

(a) Prove that $g(a)$ and $g(b)$ have the same sign.

(b) Prove that there is some number x with $a < x < b$ and $f'(x) = 0$.

(c) Now prove the same fact, even if a and b are multiple roots.

Problem 10. If f is differentiable three times and $f'(x) \neq 0$, the Schwarzian derivative of f at x is defined to be

$$\mathcal{S}f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

(a) Show that

$$\mathcal{S}(f \circ g) = [\mathcal{S}f \circ g] \cdot (g')^2 + \mathcal{S}g.$$

(b) Show that if $f(x) = \dfrac{ax + b}{cx + d}$, with $ad - bc \neq 0$, then $\mathcal{S}f = 0$. Consequently, $\mathcal{S}(f \circ g) = \mathcal{S}g$.

Problem 11. Let $a > 0$. Show that the maximum value of the function

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - a|}$$

is $(2 + a)/(1 + a)$.

Problem 12. A function f is Lipschitz of order α at x is there is a constant C such that

$$|f(x) - f(y)| \leq C|x - y|^{\alpha} \tag{*}$$

for all y in an interval around x. The function f is Lipschitz of order α on an interval if $(*)$ holds for all x and y in the interval.

(a) If f is Lipschitz of order $\alpha > 0$ at x, then f is continuous at x.

(b) If f is differentiable at x, then f is Lipschitz of order 1 at x.

(c) If f is Lipschitz of order $\alpha > 1$ at x, then f is differentiable at x and $f'(x) = 0$.

Problem 13. [Cauchy's Mean Value Theorem] Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Then there is a number x in (a, b) such that

$$\big(f(b) - f(a)\big)g'(x) = \big(g(b) - g(a)\big)f'(x).$$

(Cf. Problem 8, pg 108 of textbook for hints.)

Problem 14 (L'Hôpital's Rule)**.** Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and that $\lim_{x\to a} f'(x)/g'(x)$ exists. Then $\lim_{x\to a} f(x)/g(x)$ also exists and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

(Cf. Problem 9, pg 108 of textbook for hints.)

Problem 15. What is the largest area of any equilateral triangle enclosed in a unit square?

Problem 16. The lower right-hand corner of a page is folded over so that it just touches the left edge of the paper, as in the figure below If the width of the paper is very long, show that the minimum length of the crease is $3\sqrt{3}\alpha/4$.