Problem 1 (∗)**.** Prove that the set Q of points (x_1, x_2) in the plane $\mathbf{R}^2$ such that x_1 and x_2 are both rational numbers is disconnected.

Problem 2 (∗)**.** Prove that if A and B are connected subsets of E and $A \cap B \neq \emptyset$, then $A \cup B$ is connected.

Problem 3 (∗)**.** Prove that the unit circle $\{(x, y) \mid x^2 + y^2 = 1\}$ is connected.

Solution. Let $f(x) = \left(\dfrac{1-x^2}{1+x^2}, \dfrac{2x^2}{1+x^2}\right)$ for $x \in \mathbf{R}$. Then f is continuous and it image is the unit cricle. Since $\mathbf{R}$ is connected, so is $f(\mathbf{R})$. □

Problem 4 (∗)**.** True or False: if f is continuous and S is connected, then $f^{-1}S$ is connected.

Problem 5 (∗)**.** Prove that if f is continuous on an interval J and $f(x)$ is rational for any x in J, then f is constant on J.

Problem 6. Suppose f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$.

(i) Prove that either $f((a+b)/2) = 0$, or f has different signs at the end points $[a, (a+b)/2]$, or f has different signs at the end points of $[(a+b)/2, b]$.

If $f((a+b)/2) \neq 0$, let I_1 be one of the two intervals on which f has different signs at the endpoints. Now bisect I_1. Then either f is 0 at the midpoint, of f has opposite signs at the endpoints of one of the two intervals into which I_1 was bisected. Let I_2 be such an interval. Continue in this way to define I_n for each natural number n (unless f is 0 at some midpoint).

(ii) Prove that there is a point x in (a, b) where $f(x) = 0$.

(iii) Use the scheme described in (i) and (ii) to approximate the solution of $x^3 + 6x - 2 = 0$ with an error smaller than $1/100$. (Calculators not allowed.)

Solution. (i) If $f((a+b)/2) \neq 0$, then this number is either > 0 (and f has different signs at the endpoints of $[a, (a+b)/2]$), or < 0 (and f has different signs at the endpoints of $[(a+b)/2, b]$).

(ii) Let x be in each I_n. If $f(x) < 0$, then there is some $\delta > 0$ such that $f(y) < 0$ for all y in $[a, b]$ with $|x - y| < \delta$. Let n be such that $(b-a)/2^n < \delta$. Since Length $I_n = (b-a)/2^n$, all the points y in I_n satisfy $|x - y| \leq 1/2^n < \delta$, hence $f(y) < 0$ for all y in I_n, contradicting that f has opposite signs on the endpoints of I_n. Similarly we cannot have $f(x) > 0$, thus $f(x) = 0$.

(iii) If $f(x) = x^3 + 6x - 2$, then $f(0) = -2$ and $f(1/3) > 0$. Let $[a, b] = [0, 1/3]$ Since Length $I_n = 1/2^n$ and $3 \cdots 2^5 < 100 < 3 \cdots 2^6$, any of the endpoints of I_6 will approximate the solution of $f(x) = 0$ by less than $1/100$. □

Problem 7 (∗)**.** Find an integer n such that the polynomial equation $x^3 - x + 3 = 0$ has a solution between n and $n + 1$.

Solution. We have $(-2)^3 - (-2) + 3 = -3$ and $(-1)^3 - (-1) + 3 = 1$, so there is a solution to the equation $x^3 - x + 3$ between $n = -2$ and $n + 1 = -1$. □

Problem 8. Prove that there is some number x such that $\sin x = x - 1$.

Solution. Let $f(x) = x - 1 - \sin x$. Then $f(0) = -1 < 0$ and $f(\pi/2) = \pi/2 > 0$, so there is x in $(0, \pi/2)$ such that $f(x) = 0$, or $\sin x = x - 1$. □

Problem 9 (∗)**.** (i) Suppose that f is continuous on the interval $[0, 1]$ and that $0 \leq f(x) \leq 1$ for all x in $[0, 1]$. Prove that $f(x) = x$ for some number x in $[0, 1]$.

(ii) Let f be continuous and bounded above and below on $\mathbf{R}$. Prove that there is some number x such that $f(x) = x$.

Solution. If $f(0) = 0$ or if $f(1) = 1$, then we are done. If not, then $f(0) > 0$ and $f(1) < 1$. Let $g(x) = x - f(x)$. Then g is continuous on $[0, 1]$, $g(0) = -f(0) < 0$ and $g(1) = 1 - f(1) > 0$. By the intermediate Value Theorem there is x in $[0, 1]$ such that $g(x) = 0$, or $f(x) = x$.

(ii) If f is bounded below and above on $\mathbf{R}$, then there are numbers a and b such that $a < f(x) < b$ for all x. The continuous function $g(x) = x - f(x)$ satisfies $g(a) < 0 < g(b)$. Apply the Intermediate Value Theorem to g on $[a, b]$. □

Problem 10. One morning, exactly at sunrise, a Buddhist monk began to climb a tall mountain. The narrow path, no more than a foot or two wide, spiraled around the mountain to a glittering temple at the summit

The monk ascended the path at varying rates of speed stopping along the way to rest and to eat the dried fruit he carried with him. He reached the temple shortly before sunset. After several days of fasting and meditation he began his journey back along the same path, starting at sunrise and again walking at variable speeds with many pauses along the way. His average speed descending was, of course, greater than his average climbing speed.

Prove that there is a spot along the path that the monk will occupy on both trips at precisely the same time of day

(Martin Gardner, in *My Best Mathematical Puzzles*, Dover 1994.)

Problem 11. A function f defined on an interval I has the Intermediate Value Property on I if for any two numbers $a < b$ in I and every y strictly between $f(a)$ and $f(b)$, there is c in (a, b) such that $f(c) = y$.

(i) Prove that the function f given by $f(x) = \sin 1/x$ if $x \neq 0$ and $f(0) = 0$ has the Intermediate Value Property on the interval $[0, B]$, for any $B > 0$.

(ii) Prove that if f is increasing on the interval I and has the Intermediate Value Property on I, then f is continuous on I. (Adopting the terminology of the textbook, f is said to be increasing on I if $f(x) \leq f(y)$ whenever $x < y$ in I; it is said to be striclty increasing if $f(x) < f(y)$ whenever $x < y$.)

Solution. (i) If $0 < a < b$ are two numbers in $[0, B]$, then we apply the Intermediate Value Theorem to $f(x) = \sin 1/x$ on the interval $[a, b]$ because f is continuous on $[a, b]$.

If $0 = a < b$ and x is strictly between $0 = f(0)$ and $f(b)$, let n be a natural number such that $2/b < (2n + 1)\pi$ so that the interval $J = [2/(2n + 3)\pi, 2/(2n + 1)\pi]$ is contained in $[0, b]$. The function $f(x) = \sin 1/x$ is continuous on J and takes on the values 1 and -1 at the endpoints of J. Since $-1 \leq f(x) \leq 1$, the Intermediate Value Theorem applied to f on J implies that given any y such that $-1 < y < 1$, there is c in J such that $f(c) = y$. In particular, if y is strictly between $f(a)$ and $f(b)$, then $-1 < y < 1$ also, and c in J satisfies $0 = a < c < b$, as desired.

(ii) Suppose that there is a in I where f fails to be continuous. Then there is a sequence (x_n) in I such that $x_n \to a$ but $f(x_n)$ does not converge to $f(a)$. We may assume, by taking a subsequence if necessary, that x_n increases (or decreases) to a. Then $f(x_n)$ is non decreasing and bounded above by $f(a)$, thus it converges to a number p with $p < f(a)$. Let q be a number such that $p < q < f(a)$. For each x_n we have $f(x_n) \leq p < q$, so the intermediate value property of f on the interval $[x_n, a]$ implies the existence of y_n in (x_n, a) such that $f(y_n) = q$. The sequence (y_n) converges to a and $f(y_n) = q$ for all n. Since x_n also converges to a, given n there is m such that $y_n < x_m$, but $f(y_n) = q > p \geq f(x_m)$, contradicting that f is non decreasing.

□

Problem 12 (∗). (i) Prove that $f(x) = x^2$ is not uniformly continuous.

(ii) Prove that $f(x) = \sqrt{x}$ is uniformly continuous.

Problem 13 (∗). (i) Prove that if f and g are uniformly continuous on E, then so is $f + g$.

(ii) Prove that if f and g are uniformly continuous and bounded on E, then fg is uniformly continuous on E.

(iii) Show that the conclusion in (ii) above does not hold if one of the function f or g is not bounded.

Solution. (i) Given $\varepsilon > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that for any x, y in E, if $|x - y| < \delta_1$, then $|f(x) - f(y)| < \varepsilon/2$ and if $|x - y| < \delta_2$, then $|g(x) - g(y)| < \varepsilon/2$. Therefore, if $|x - y| < min\{\delta_1, \delta_2\}$, then

$$|(f + g)(x) - (f + g)(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

as desired.

(ii) Because f and g are bounded, there is $M > 0$ be such that $|f(x)| < M$ and $|g(x)| < M$ for all x in E. Because f and g are uniformly continuous on E, given $\varepsilon > 0$, there is $\delta > 0$ such that if x and y are in E and $|x - y| < \delta$, then

$|f(x) - f(y)| < \varepsilon/2M$ and $|g(x) - g(y)| < \varepsilon/2M$. Then

$$\begin{aligned}
|f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)| && \text{(Add and subtract } f(y)g(x)\text{)} \\
&\leq |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - f(y)g(y)| && \text{(Triangle Inequality)} \\
&\leq |g(x)||f(x) - f(y)| + |f(y)||g(x) - g(y)| && \text{(Extract common factors)} \\
&\leq M|f(x) - f(y)| + M|g(x) - g(y)| && (f \text{ and } g \text{ bounded by } M) \\
&< M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} && \text{(Uniform continuity)} \\
&= \varepsilon
\end{aligned}$$

(iii) Let $f(x) = x$ and $g(x) = \sin x$. Buth functions are uniformly continuous on $\mathbf{R}$, but the product $f \cdot g$ is not. □

Problem 14 ($*$)**.** Let $f : E \subset \mathbf{R} \to \mathbf{R}$ be uniformly continuous. Prove that if (x_n) is a Cauchy sequence in E, then $(f(x_n))$ is also a Cauchy sequence. Show by counterexample that *uniformly* is necessary.

Solution. If f is uniformly continuous on E, then given $\varepsilon > 0$ there is $\delta > 0$ such that if x, y are in E and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Let (x_n) be a Cauchy sequence in E. Then given $\delta > 0$ there is N such that if $p, q > N$, then $|x_p - x_q| < \delta$, and thus $|f(x_p) - f(x_q)| < \varepsilon$, implying that $(f(x_n))$ is a Cauchy sequence. □

Note. A Cauchy sequence in E is also a Cauchy sequence in $\mathbf{R}$, and thus it converges. But the key observation is that it may not converge to a point in E (it will converge to a point in $\overline{E}$).

Problem 15. We proved in class that a continuous real valued function on a compact set attains a maximum value. Find the maximum value of $f(x) = x^3 - 9x$ in the interval $[-3, 3]$. **Note:** No derivatives!