Problem 1 ($*$). Prove that the set Q of points (x_1, x_2) in the plane $\mathbf{R}^2$ such that x_1 and x_2 are both rational numbers is disconnected.

Problem 2 ($*$). Prove that if A and B are connected subsets of E and $A \cap B \neq \emptyset$, then $A \cup B$ is connected.

Problem 3 ($*$). Prove that the unit circle $\{(x, y) \mid x^2 + y^2 = 1\}$ is connected.

Problem 4 ($*$). True or False: if f is continuous and S is connected, then $f^{-1}S$ is connected.

Problem 5 ($*$). Prove that if f is continuous on an interval J and $f(x)$ is rational for any x in J, then f is constant on J.

Problem 6. Suppose f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$.

(i) Prove that either $f((a+b)/2) = 0$, or f has different signs at the end points $[a, (a+b)/2]$, or f has different signs at the end points of $[(a+b)/2, b]$.

If $f((a+b)/2) \neq 0$, let I_1 be one of the two intervals on which f has different signs at the endpoints. Now bisect I_1. Then either f is 0 at the midpoint, of f has opposite signs at the endpoints of one of the two intervals into which I_1 was bisected. Let I_2 be such an interval. Continue in this way to define I_n for each natural number n (unless f is 0 at some midpoint).

(ii) Prove that there is a point x in (a, b) where $f(x) = 0$.

(iii) Use the scheme described in (i) and (ii) to approximate the solution of $x^3 + 6x - 2 = 0$ with an error smaller than $1/100$. (Calculators not allowed.)

Problem 7 ($*$). Find an integer n such that the polynomial equation $x^3 - x + 3 = 0$ has a solution between n and $n + 1$.

Problem 8. Prove that there is some number x such that $\sin x = x - 1$.

Problem 9 ($*$). (i) Suppose that f is continuous on the interval $[0, 1]$ and that $0 \leq f(x) \leq 1$ for all x in $[0, 1]$. Prove that $f(x) = x$ for some number x in $[0, 1]$.

(ii) Let f be continuous and bounded above and below on $\mathbf{R}$. Prove that there is some number x such that $f(x) = x$.

Problem 10. One morning, exactly at sunrise, a Buddhist monk began to climb a tall mountain. The narrow path, no more than a foot or two wide, spiraled around the mountain to a glittering temple at the summit

The monk ascended the path at varying rates of speed stopping along the way to rest and to eat the dried fruit he carried with him. He reached the temple shortly before sunset. After several days of fasting and meditation he began his journey back along the same path, starting at sunrise and again walking at variable speeds with many pauses along the way. His average speed descending was, of course, greater than his average climbing speed.

Prove that there is a spot along the path that the monk will occupy on both trips at precisely the same time of day

(Martin Gardner, in *My Best Mathematical Puzzles*, Dover 1994.)

Problem 11. A function f defined on an interval I has the Intermediate Value Property on I if for any two numbers $a < b$ in I and every y strictly between $f(a)$ and $f(b)$, there is c in (a, b) such that $f(c) = y$.

(i) Prove that the function f given by $f(x) = \sin 1/x$ if $x \neq 0$ and $f(0) = 0$ has the Intermediate Value Property on the interval $[0, B]$, for any $B > 0$.

(ii) Prove that if f is non decreasing on the interval I and has the Intermediate Value Property on I, then f is continuous on I. (Recall that f is said to be non decreasing on I if $f(x) \leq f(y)$ whenever $x < y$ in I.)

Problem 12 ($*$). (i) Prove that $f(x) = x^2$ is not uniformly continuous.

(ii) Prove that $f(x) = \sqrt{x}$ is uniformly continuous.

Problem 13 ($*$). (i) Prove that if f and g are uniformly continuous on E, then so is $f + g$.

(ii) Prove that if f and g are uniformly continuous and bounded on E, then fg is uniformly continuous on E.

(iii) Show that the conclusion in (ii) above does not hold if one of the function f or g is not bounded.

Problem 14 ($*$)**.** Let $f : E \subset \mathbf{R} \to \mathbf{R}$ be uniformly continuous. Prove that if (x_n) is a Cauchy sequence in E, then $(f(x_n))$ is also a Cauchy sequence. Show by counterexample that *uniformly* is necessary.

Problem 15. We proved in class that a continuous real valued function on a compact set attains a maximum value. Find the maximum value of $f(x) = x^3 - 9x$ in the interval $[-3, 3]$. **Note:** No derivatives!