Problem 1. (a) Define the concept "f is a continuous function at the point p."

(b) Let $f : \mathbf{R} \to \mathbf{R}$ be the function given by $f(x) = x$ if x is rational, and $f(x) = -x$ if x is irrational. Prove that f is continuous only at $p = 0$.

Problem 2 (IV.1)**.** Discuss the continuity of the function $f : \mathbf{R} \to \mathbf{R}$ if

(a) $f(x) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$

(b) $f(x) = \begin{cases} x^2, & x \neq 0 \\ 1, & x = 0 \end{cases}$

(c) $f(x) = \sqrt{|x|}$

Problem 3 (IV.10)**.** Discuss the continuity of the function $f : \mathbf{R}^2 \to \mathbf{R}$ if

(a) $f(x,y) = \begin{cases} \dfrac{1}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(b) $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(c) $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Problem 4. For a non-empty subset S of $\mathbf{R}^k$, let d_S be the function given by $d_S(x) = \text{g.l.b.}\{|x-y| \mid y \in S\}$, where $|x-y|$ is the distance between the vectors x and y. Prove that d_S is continuous on $\mathbf{R}^k$. (Consider first the case in which $S = \{x_0\}$, a singleton.)

Solution. Let S be a non-empty subset of E. Define d_S to be the function

$$d_S(x) = \text{g.l.b.}\{|x-y| \mid y \in S\}.$$

This is well defined because the set of numbers $\{|x-z| \mid z \in S\}$ is non–empty and bounded below by 0 (so it has a greatest lower bound).

For any $x \in E$ and any $z \in S$, we have, $d_S(x) \leq |x-z|$, and by the triangle inequality, for any y in E:

$$d_S(x) \leq |x-z| \leq |x-y| + |y-z|$$

or

$$d_S(x) - |x-y| \leq |y-z|$$

Problem 5. (a) Give an example of a continuous function on a closed set $E \subset \mathbf{R}$ that has no maximum.

(b) Give an example of a continuous function on a bounded set $F \subset \mathbf{R}$ that has no maximum.

Problem 6 (IV.7)**.** Let I be an interval in $\mathbf{R}$ and let $a \in I$. If f is a function whose domain contains $I \setminus \{a\}$, define

$$\lim_{x\to a+} f(x) = \lim_{x\to a} f_+(x)$$

where f_+ is the function with domain $I \cap (a, \infty)$ given by $f_+(x) = f(x)$. Similarly, define

$$\lim_{x\to a-} f(x) = \lim_{x\to a} f_-(x)$$

where f_- is the function with domain $I \cap (-\infty, a)$ given by $f_+(x) = f(x)$. Prove that $\lim_{x\to a} f(x)$ exists if and only if both $\lim_{x\to a+} f(x)$ and $\lim_{x\to a-} f(x)$ exist and are equal.

Problem 7 (IV.8)**.** Let f be a real valued function defined on (a, ∞), where $a > 0$ is some positive real number. Let $\lim_{x \to \infty+} f(x)$ be given by

$$\lim_{x \to \infty+} f(x) = \lim_{y \to 0} g(y),$$

where $g : (0, 1/a) \to \mathbf{R}$ is given by $g(y) = f(1/y)$, if this latter limit exists.

Prove that $\lim_{x \to \infty+} f(x)$ exists if and only if for any $\varepsilon > 0$ there exists a number $N \geq a$ such that $|f(x) - f(y)| < \varepsilon$ if $x, y > N$.

Problem 8 (IV.4)**.** Let I, J be open intervals in $\mathbf{R}$ and let $f : I \to J$ be a function that is strictly increasing (that is, if $x < y$, then $f(x) < f(y)$) and surjective. Prove that f is continuous.

Problem 9. Suppose that f is continuous on an open set S. Prove that if p is in S, then there exists $r > 0$ such that $f(B(p, r))$ is a bounded set.

Problem 10. True or false:

(a) f is continuous if and only if f takes convergent sequences to convergent sequences (that is, if $p_1, p_2, p_3, \cdots$ is a sequence in dom(f) that converges to a point in dom(f), then $f(p_1), f(p_2), f(p_3), \cdots$ converges).

(b) f is continuous if and only if f takes Cauchy sequences to Cauchy sequences (that is, if $p_1, p_2, p_3, \cdots$ is a Cauchy sequence in dom(f), then $f(p_1), f(p_2), f(p_3), \cdots$ is a Cauchy sequence).

Problem 11. (a) Consider a sequence of closed intervals $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, $\ldots$. Suppose that $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for all n. Prove that there exists a point x that is in every I_n.

(b) Prove that if Length $I_n \to 0$, then the point x in (a) is unique.

(c) Show that this conclusion in Part (i) is false if we consider open intervals instead of closed intervals. Is it true if we consider open and bounded intervals?