

Note. Unless otherwise indicated or implied by the context, all functions f , g , and so on in the problems below are assumed to be real valued functions of a real variable.

Problem 1. Find the domain of the functions defined by the following formulas.

- (a) $f(x) = \sqrt{1-x^2}$.
 (b) $f(x) = \sqrt{1-\sqrt{1-x^2}}$.
 (c) $f(x) = \frac{1}{x-1} + \frac{1}{x-2}$.
 (d) $f(x) = \sqrt{1-x^2} + \sqrt{x^2-1}$.
 (e) $f(x) = \sqrt{1-x} + \sqrt{x-2}$.

Solution. (b) The domain of $f(x) = \sqrt{1-\sqrt{1-x^2}}$ is the set of all x such that $x^2 \leq 1$ (inside square root) and $\sqrt{1-x^2} \leq 1$ (outside square root). If $x^2 \leq 1$, then $0 \leq 1-x^2 \leq 1$, and then $\sqrt{1-x^2} \leq 1$. Therefore the second inequality is a consequence of the first. The domain of f is then the interval $[-1, 1]$

- (d) The domain of $f(x) = \sqrt{1-x^2} + \sqrt{x^2-1}$ is the set of all x such that $1 \leq x^2$ (first square root) and $x^2 \leq 1$ (second square root). Only the numbers $x = 1$ and $x = -1$ satisfy both inequalities. □

Problem 2. Find the following limits.

- (a) $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x + 1}$.
 (b) $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$.
 (c) $\lim_{x \rightarrow 3} \frac{x^3 - 8}{x - 2}$.
 (d) $\lim_{x \rightarrow y} \frac{x^n - y^n}{x - y}$.
 (e) $\lim_{y \rightarrow x} \frac{x^n - y^n}{x - y}$.
 (f) $\lim_{h \rightarrow 0} \frac{\sqrt{a+h} - \sqrt{a}}{h}$.

Problem 3. In each of the following cases, find a δ such that $|f(x) - L| < \varepsilon$ for all x satisfying $0 < |x - a| < \delta$.

- (a) $f(x) = x^4$; $L = a^4$.
 (b) $f(x) = 1/x$; $a = 1$, $L = 1$.
 (c) $f(x) = x^4 + (1/x)$; $a = 1$, $L = 2$.
 (d) $f(x) = \frac{x}{1 + \sin^2 x}$; $a = 0$, $L = 0$.
 (e) $f(x) = \sqrt{|x|}$; $a = 0$, $L = 0$.

- (f) $f(x) = \sqrt{x}$; $a = 1$, $L = 1$.

Proof. (b) If $0 < |x - 1| < \min\{1/2, \varepsilon/2\}$, then $|x| - 1 \leq |x - 1| \leq \frac{1}{2}$, and thus $|x| > \frac{1}{2}$. Therefore, taking $\delta = \min\{1/2, \varepsilon/2\}$, if $0 < |x - 1| < \delta$, then

$$\begin{aligned} \left| \frac{1}{x} - 1 \right| &= \frac{|1-x|}{|x|} \\ &\leq \frac{\varepsilon/2}{1/2} = \varepsilon \end{aligned}$$

- (f) Note that $|x-1| = |\sqrt{x}-1||\sqrt{x}+1|$ and that $|\sqrt{x}+1| \geq 1$ for all x . Given $\varepsilon > 0$, take $\delta = \varepsilon$. If $|x - 1| < \delta$, then

$$|\sqrt{x} - 1| = \frac{|x - 1|}{|\sqrt{x} + 1|} \leq \varepsilon.$$

□

Problem 4. Suppose that $f(x) \leq g(x)$ for all x .

- (a) Prove that $\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$ provided that these limits exist.
 (b) If $f(x) < g(x)$ for all x , does it necessarily follow that $\lim_{x \rightarrow a} f(x) < \lim_{x \rightarrow a} g(x)$? (Explain.)

Solution. (a) Let $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$. Suppose that it is false that $L \leq M$. Then take $\varepsilon = (L - M) > 0$ and obtain δ so that, for all x , if $0 < |x - a| < \delta$, then

$$|f(x) - L| < \varepsilon \quad \text{and} \quad |g(x) - M| < \varepsilon.$$

It follows that for all x , if $0 < |x - a| < \delta$, then

$$-\frac{L - M}{2} < L - f(x) < \frac{L - M}{2},$$

and

$$-\frac{L - M}{2} < g(x) - M < \frac{L - M}{2}.$$

Combining these inequalities

$$-(L - M) < g(x) - f(x) + (L - M) < L - M,$$

or

$$-2(L - M) < g(x) - f(x) < 0,$$

which contradicts the hypothesis $g(x) \geq f(x)$.

- (c) Take $f(x) = 0$ if $x \neq 0$, $f(0) = -1$, and take $g(x) = x^2$. Then $g(x) > f(x)$ for all x , but

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} g(x) = 0.$$

□

Problem 5. (a) If $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ do not exist, can $\lim_{x \rightarrow a} [f(x) + g(x)]$ or $\lim_{x \rightarrow a} (f \cdot g)(x)$ exist?

- (b) If $\lim_{x \rightarrow a} f(x)$ exists and $\lim_{x \rightarrow a} [f(x) + g(x)]$ exists, must $\lim_{x \rightarrow a} g(x)$ exist?
- (c) If $\lim_{x \rightarrow a} f(x)$ exists and $\lim_{x \rightarrow a} g(x)$ does not exist, can $\lim_{x \rightarrow a} [f(x) + g(x)]$ exist?
- (d) If $\lim_{x \rightarrow a} f(x)$ exists and $\lim_{x \rightarrow a} f(x)g(x)$ exists, does it follow that $\lim_{x \rightarrow a} g(x)$ exists?

Solution. (a) Yes. For the sum, let $f(x) = 1/(x - a)$ and $g(x) = -1/(x - a)$, for $x \neq a$. Neither f nor g have limit when $x \rightarrow a$. But $f + g = 0$ which has limit 0 when $x \rightarrow a$. (If you want to have f and g defined at a , then make $f(a) = g(a) = 0$; this is irrelevant for the existence of limit at a .)

For the product, let

$$f(x) = \begin{cases} -1, & x < a \\ 1, & x \geq a \end{cases}$$

and $g(x) = f(x)$. Then f and g have no limit when $x \rightarrow a$; but $f \cdot g(x) = 1$ for all x , which has limit equal to 1 when $x \rightarrow a$.

- (b) Yes, applying the algebra of limits (Corollary on page 76 of textbook) to $f + g$ and $-f$, because $f(x) + g(x) - f(x) = g(x)$.
- (c) No, by (b).
- (d) Not necessarily. From page 91-92, take $f(x) = x$, $g(x) = \sin \frac{1}{x}$. Then $\lim_{x \rightarrow 0} f(x) = 0$ and $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$ (page 92) both exist, but $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ does not exist. Another example: take $f(x) = x - a$ and $g(x) = 1/(x - a)$ for $x \neq a$. Then $\lim_{x \rightarrow a} f(x) = 0$, $\lim_{x \rightarrow a} f(x)g(x) = 1$, but $\lim_{x \rightarrow a} g(x)$ does not exist.

□

Problem 6. (a) Suppose that $\lim_{x \rightarrow 0} f(x)$ exists. Prove that if g is any function such that $\lim_{x \rightarrow 0} g(x)$ does not exist, then $\lim_{x \rightarrow 0} [f(x) + g(x)]$ also does not exist.

- (b) Suppose that $\lim_{x \rightarrow 0} f(x)$ exists and is $\neq 0$. Prove that if g is any function such that $\lim_{x \rightarrow 0} g(x)$ does not exist, then $\lim_{x \rightarrow 0} [f(x) \cdot g(x)]$ also does not exist.

Problem 7. Define $\lim_{x \rightarrow a} f(x) = \infty$ to mean that for every N there is a $\delta > 0$ such that, for all x , if $0 < |x - a| < \delta$, then $f(x) > N$. Prove that $\lim_{x \rightarrow 3} \frac{1}{(x - 3)^2} = \infty$.

Solution. Given N , let $\delta = \frac{1}{\sqrt{1 + |N|}}$. If $0 < |x - 3| < \delta$, then

$$\frac{1}{(x - 3)^2} > \frac{1}{\delta^2} = 1 + |N| > N.$$

(The reason for taking such δ , instead of the obvious choice $1/\sqrt{N}$, is that we don't know that the given N is ≥ 0 .) □

Problem 8. Prove that if $\lim_{x \rightarrow 0} f(x)/x = L$ and $b \neq 0$, then $\lim_{x \rightarrow 0} f(bx)/x = bL$.

Solution. We must show that given $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x| < \delta$, then

$$\left| \frac{f(bx)}{x} - bL \right| < \varepsilon.$$

Note that

$$\left| \frac{f(bx)}{x} - bL \right| = |b| \left| \frac{f(bx)}{bx} - L \right|.$$

Because $\lim_{x \rightarrow 0} f(x)/x = L$, for $\varepsilon_1 = \varepsilon/|b| > 0$ there exists $\delta_1 > 0$ such that if $0 < |x| < \delta_1$, then

$$\left| \frac{f(x)}{x} - L \right| < \varepsilon_1 = \frac{\varepsilon}{|b|}.$$

Take $\delta = \delta_1/|b|$. If $0 < |x| < \delta$, then $0 < |bx| = |b||x| < |b|\delta = \delta_1$, and so

$$\begin{aligned} \left| \frac{f(bx)}{x} - bL \right| &= \left| b \left(\frac{f(bx)}{bx} - L \right) \right| \\ &= |b| \left| \frac{f(bx)}{bx} - L \right| \\ &< |b| \varepsilon_1 \quad (\text{because } 0 < |bx| < \delta_1) \\ &= |b| \varepsilon / |b| \\ &= \varepsilon. \end{aligned}$$

□

Problem 9. (You must show that your example satisfies the required property.)

- (a) Find a function $f : \mathbf{R} \rightarrow \mathbf{R}$ which is continuous except at the integers.
- (b) Find a function $f : \mathbf{R} \rightarrow \mathbf{R}$ which is continuous only at 0.
- (c) For each number p , find a function which is continuous at p , but not at any other points.
- (d) Find a function $f : \mathbf{R} \rightarrow \mathbf{R}$ which is discontinuous at 1, 1/2, 1/3, ... but continuous at all other points.
- (e) Find a function $f : \mathbf{R} \rightarrow \mathbf{R}$ which is discontinuous at 1, 1/2, 1/3, ..., and at 0, but continuous at all other points.

Problem 10. (a) Suppose that f is a function satisfying $|f(x)| \leq |x|$ for all x . Show that f is continuous at 0.

- (b) Give an example of such a function f which is not continuous at any $a \neq 0$.

- (c) Suppose that g is continuous at 0, $g(0) = 0$, and f is a function satisfying $|f(x)| \leq |g(x)|$ for all x . Show that f is continuous at 0.

Solution. (a) If $|f(x)| \leq |x|$ for all x , then $|f(0)| \leq |0| = 0$, so $f(0) = 0$. Given $\varepsilon > 0$, let $\delta = \varepsilon$. If $|x| < \delta$, then

$$||f(x)| - |f(0)|| = |f(x)| \leq |x| < \delta = \varepsilon.$$

- (b) An example is given on page 114, namely,

$$f(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

□

Problem 11. (a) Prove that if f is continuous at p , then so is $|f|$.

- (b) Give an example of a function f such that f is continuous nowhere, but $|f|$ is continuous everywhere.

- (c) Prove that if f and g are continuous, so are $\max\{f, g\}$ and $\min\{f, g\}$

Problem 12. (a) Prove that if f and g are continuous, then so is $f + g$.

- (b) Prove that if f and g are continuous, then so is $f \cdot g$.

- (c) Give an example of two functions f and g , both discontinuous at 0, whose sum is continuous at 0.

- (d) Give an example of two functions f and g , both discontinuous at 0, whose product is continuous at 0.

Solution. (c) Let $f(x) = -1$ if x is rational and $f(x) = 1$ if x is not rational. Let $g = f$. Then $(f + g)(x) = 0$ for all x and $(f \cdot g)(x) = 1$ for all x .

□

Problem 13. (a) Suppose that $\lim_{x \rightarrow p} f(x) = a \neq 0$. Prove

$$\text{that } \lim_{x \rightarrow p} \frac{1}{f}(x) = \frac{1}{a}.$$

- (b) Let $r(x) = \frac{p(x)}{q(x)}$ be a rational function, where $p(x)$ and $q(x)$ are polynomials in x . Prove that r is continuous at all x such that $q(x) \neq 0$.

Solution. Use algebra of limits (Corollary on page 76). □