Note. Unless otherwise indicated or implied by the context, all functiuons f, g, and so on in the problems below are assumed to be real valued functions of a real variable.

Problem 1. Find the domain of the functions defined by the following formulas.

(a) $f(x) = \sqrt{1-x^2}$.

(b) $f(x) = \sqrt{1-\sqrt{1-x^2}}$.

(c) $f(x) = \dfrac{1}{x-1} + \dfrac{1}{x-2}$.

(d) $f(x) = \sqrt{1-x^2} + \sqrt{x^2-1}$.

(e) $f(x) = \sqrt{1-x} + \sqrt{x-2}$.

Problem 2. Find the following limits.

(a) $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x+1}$.

(b) $\displaystyle\lim_{x\to 2} \frac{x^3-8}{x-2}$.

(c) $\displaystyle\lim_{x\to 3} \frac{x^3-8}{x-2}$.

(d) $\displaystyle\lim_{x\to y} \frac{x^n-y^n}{x-y}$.

(e) $\displaystyle\lim_{y\to x} \frac{x^n-y^n}{x-y}$.

(f) $\displaystyle\lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h}$.

Problem 3. In each of the following cases, find a δ such that $|f(x)-L| < \varepsilon$ for all x satisfying $0 < |x-a| < \delta$.

(a) $f(x) = x^4$; $L = a^4$.

(b) $f(x) = 1/x$; $a = 1$, $L = 1$.

(c) $f(x) = x^4 + (1/x)$; $a = 1$, $L = 2$.

(d) $f(x) = \dfrac{x}{1+\sin^2 x}$; $a = 0$, $L = 0$.

(e) $f(x) = \sqrt{|x|}$; $a = 0$, $L = 0$.

(f) $f(x) = \sqrt{x}$; $a = 1$, $L = 1$.

Problem 4. Suppose that $f(x) \leq g(x)$ for all x.

(a) Prove that $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$ provided that these limits exist.

(b) If $f(x) < g(x)$ for all x, does it necessarily follow that $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$? (Explain.)

Problem 5. (a) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist, can $\lim_{x\to a}\big[f(x)+g(x)\big]$ or $\lim_{x\to a}(f\cdot g)(x)$ exist?

(b) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a}\big[f(x)+g(x)\big]$ exists, must $\lim_{x\to a} g(x)$ exist?

(c) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, can $\lim_{x\to a} \big[f(x)+g(x)\big]$ exist?

(d) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x)g(x)$ exists, does it follow that $\lim_{x\to a} g(x)$ exists?

Problem 6. (a) Suppose that $\lim_{x\to 0} f(x)$ exists. Prove that if g is any function such that $\lim_{x\to 0} g(x)$ does not exists, then $\lim_{x\to 0} [f(x)+g(x)]$ also does not exist.

(b) Suppose that $\lim_{x\to 0} f(x)$ exists and is $\neq 0$. Prove that if g is any function such that $\lim_{x\to 0} g(x)$ does not exists, then $\lim_{x\to 0} [f(x)\cdot g(x)]$ also does not exist.

Problem 7. Define $\lim_{x\to a} f(x) = \infty$ to mean that for every N there is a $\delta > 0$ such that, for all x, if $0 < |x-a| < \delta$, then $f(x) > N$. Prove that $\lim_{x\to 3} \dfrac{1}{(x-3)^2} = \infty$.

Problem 8. Prove that if $\lim_{x\to 0} f(x)/x = L$ and $b \neq 0$, then $\lim_{x\to 0} f(bx)/x = bL$.

Problem 9. (You must show that your example satisfies the required property.)

(a) Find a function $f : \mathbf{R} \to \mathbf{R}$ which is continuous except at the integers.

(b) Find a function $f : \mathbf{R} \to \mathbf{R}$ which is continuous only at 0.

(c) For each number p, find a function which is continuous at p, but not at any other points.

(d) Find a function $f : \mathbf{R} \to \mathbf{R}$ which is discontinuous at $1, 1/2, 1/3, \ldots$ but continuous at all other points.

(e) Find a function $f : \mathbf{R} \to \mathbf{R}$ which is discontinuous at $1, 1/2, 1/3, \ldots$, and at 0, but continuous at all other points.

Problem 10. (a) Suppose that f is a function satisfying $|f(x)| \leq |x|$ for all x. Show that f is continuous at 0.

(b) Give an example of such a function f which is not continuous at any $a \neq 0$.

(c) Suppose that g is continuous at 0, $g(0) = 0$, and f is a function satisfying $|f(x)| \leq |g(x)|$ for all x. Show that f is continuous at 0.

Problem 11. (a) Prove that if f is continuous at p, then so is $|f|$.

(b) Give an example of a function f such that f is continuous nowhere, but $|f|$ is continuous everywhere.

(c) Prove that if f and g are continuous, so are $\max\{f, g\}$ and $\min\{f, g\}$

Problem 12. (a) Prove that if f and g are continuous, then so is $f + g$.

(ii) Prove that if f and g are continuous, then so is $f \cdot g$.

(b) Give an example of two functions f and g, both discontinuous at 0, whose sum is continuous at 0.

(c) Give an example of two functions f and g, both discontinuous at 0, whose product is continuous at 0.

Problem 13. (a) Suppose that $\lim_{x\to p} f(x) = a \neq 0$. Prove that $\lim_{x\to p} \dfrac{1}{f}(x) = \dfrac{1}{a}$.

(b) Let $r(x) = \dfrac{p(x)}{q(x)}$ be a rational function, where $p(x)$ and $q(x)$ are polynomials in x. Prove that r is continuous at all x such that $q(x) \neq 0$.