

Problem 1. Let S be a nonempty subset of real number that is bounded above but has no greatest element. Prove that l. u. b. S is a cluster point of S .

Problem 2. A sequence p_1, p_2, p_3, \dots is called a Cauchy sequence if, for every $\varepsilon > 0$, there is a natural number N such that whenever $n, m > N$, the distance $d(p_n, p_m) < \varepsilon$.

(a) Prove that a convergent sequence is a Cauchy sequence.

(b) Prove that a Cauchy sequence is bounded.

Proof. Solution(a) Suppose that $a_n \rightarrow l$. Given $\varepsilon > 0$ there is a natural number N such that if $n > N$, then $|a_n - l| < \varepsilon/2$. Therefore, if $p, q > N$,

$$|a_p - a_q| = |a_p - l + l - a_q| \leq |a_p - l| + |a_q - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

thus showing that (a_n) is Cauchy. \square

Problem 3. Prove the following:

(a) A subsequence of a Cauchy sequence is a Cauchy sequence.

(b) If a subsequence of a Cauchy sequence converges, then the whole sequence also converges.

Problem 4. Write down in all detail (and I cannot stress enough the relevance of doing this in all detail) that any Cauchy sequence in \mathbf{R}^k is convergent.

Problem 5. Let S be a subset of $E = \mathbf{R}^k$ and let p be a point of E (p may or may not be in S). Prove that the following are equivalent:

(a) Any ball $B(p, r)$ contains a point q of S such that $q \neq p$. That is, for any $r > 0$, the intersection $(B(p, r) \setminus \{p\}) \cap S \neq \emptyset$.

(b) Any ball $B(p, r)$ contains infinitely many points of S .

(c) There is a sequence q_1, q_2, q_3, \dots of points of S such that $q_n \neq p$ and $\lim_{n \rightarrow \infty} q_n = p$.

Problem 6. Prove that a set $S \subset \mathbf{R}^k$ is bounded if and only if every sequence of points in S has a convergent subsequence.

Solution. A sequence (x_n) is bounded if there is a number M such that $|x_n| \leq M$ for all n .

Assume that $S \subset \mathbf{R}$ is bounded. If (x_n) is a sequence in S , then (x_n) is bounded and thus it has a convergent subsequence.

Assume that S is not bounded. Then for any integer n there is an x_n in S such that $|x_n| > n$. Since any convergent sequence is bounded, the sequence (x_n) cannot have a convergent subsequence. \square

Problem 7. Prove the following:

(a) The intersection of an arbitrary family of compact sets is compact.

(b) The union of finitely many compact sets is compact.

Problem 8. Give an example of open sets $U_1 \supset U_2 \supset \dots$ in E such that the intersection $\bigcap_{n=1}^{\infty} U_n$ is closed and nonempty.

Solution. Let $U_n = (-1/n, 1/n)$. Then the intersection $\bigcap_{n=1}^{\infty} U_n = \{0\}$. \square

Problem 9. Give an example of closed sets $C_1 \supset C_2 \supset \dots$ in E such that the intersection $\bigcap_{n=1}^{\infty} C_n$ is empty.

Solution. Let $C_n = [n, \infty)$, $n = 1, 2, 3, \dots$ \square

Problem 10. (a) Prove that any intersection of compact sets is a compact set.

(b) Prove that a union of finitely many compact sets is a compact set.

Solution. (a) Let $\{K_i\}_{i \in I}$ be a family of compact sets, and let $K = \bigcap_{i \in I} K_i$ denote their intersection. We'll show that K is compact by showing that it is closed and bounded. Each K_i is bounded (because it is compact) and $K \subset K_i$ (for all i) so K must be bounded, as any bound for K_i will also be a bound for K .

The set K is also closed because the intersection of closed sets is a closed set.

(b) Suppose that K_1 and K_2 are compact, and let $K = K_1 \cup K_2$ be their union. Let (x_n) be a sequence in K . Each x_n is in one of the two sets K_1 or K_2 (it could be in both), so it follows that there is a subsequence (x_{n_m}) of (x_n) where all the terms x_{n_m} , $m = 1, 2, \dots$, are in the same K_i , $i = 1$ or 2 . Since K_i is compact, this (sub)sequence (x_{n_m}) has a subsequence $(x_{n_{m_l}})$ which converges to a point x in K_i . But $(x_{n_{m_l}})$ is also a subsequence of the original sequence (x_n) , and its limit x is in $K \supset K_i$.

This proves that the union of two compact sets is compact. For finite unions, the proof proceeds by induction on the number of sets. Suppose that you have proved that the union of $< n$ compact sets is a compact. If K_1, \dots, K_n is a collection of n compact sets, then their union can be written as $K = K_1 \cup (K_2 \cup \dots \cup K_n)$, the union of two compact sets, hence compact. \square

Problem 11. Prove or give a counterexample:

(a) The union of infinitely many compact sets is compact.

(b) A non-empty subset S of real numbers which has both a largest and a smallest element is compact.

Solution. (a) False. We know that \mathbf{R} is not compact (for example, because the sequence of natural numbers (n) has no convergent subsequence, or because \mathbf{R} is not bounded), but we can write $\mathbf{R} = \bigcup_{n=1}^{\infty} [-n, n]$, where each interval $[-n, n]$ is compact because it is closed and bounded.

(b) False. The set $S = [-1, 0) \cup (0, 1]$ has a largest element, namely 1, and a smallest element, namely -1, but it is not compact because, for example, the sequence $(1/n)$ is in S but has no subsequence which converges to a point in S . \square

Problem 12. (a) Consider a sequence of closed intervals $I_1 = [a_1, b_1], I_2 = [a_2, b_2], \dots$. Suppose that $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for all n . Prove that there is a point x which is in every I_n .

(b) Prove that if $\text{Length } I_n \rightarrow 0$, then the point x in (a) is unique.

(c) Show that this conclusion in (a) is false if we consider open intervals instead of closed intervals. Is it true if we consider open and bounded intervals?

Solution. (a) The sequence of left endpoints (a_n) is non-decreasing and bounded above (by b_1), so it converges to $a = \text{l.u.b.}\{a_n \mid n \text{ in } \mathbf{N}\}$. Similarly the sequence of right endpoints (b_n) converges to $b = \text{g.l.b.}\{b_n \mid n \text{ in } \mathbf{N}\}$. Since $a_n \leq b_m$ (because $a_n \leq a_{n+m} \leq b_{n+m} \leq b_m$), we have $a \leq b$. If x is any number such that $a \leq x \leq b$, then $a_n \leq x \leq b_n$ for all n , which means that x is in I_n for all n .

(b) If $\text{Length } I_n = b_n - a_n \rightarrow 0$, then $a = b$.

(c) $I_n = (0, 1/n)$.

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