Problem 1. Let S be a nonempty subset of real number that is bounded above but has no greatest element. Prove that $\text{l.u.b.}\, S$ is a cluster point of S.

Problem 2. A sequence $p_1, p_2, p_3, \cdots$ is called a Cauchy sequence if, for every $\varepsilon > 0$, there is a natural number N such that whenever $n, m > N$, the distance $d(p_n, p_m) < \varepsilon$.

(a) Prove that a convergent sequence is a Cauchy sequence.

(b) Prove that a Cauchy sequence is bounded.

Problem 3. Prove the following:

(a) A subsequence of a Cauchy sequence is a Cauchy sequence.

(b) If a subsequence of a Cauchy sequence converges, then the whole sequence also converges.

Problem 4. Write down in all detail (and I cannot stress enough the relevance of doing this in all detail) that any Cauchy sequence in $\mathbf{R}^k$ is convergent.

Problem 5. Let S be a subset of $E = \mathbf{R}^k$ and let p be a point of E (p may or may not be in S). Prove that the following are equivalent:

(a) Any ball $B(p, r)$ contains a point q of S such that $q \neq p$. That is, for any $r > 0$, the intersection $(B(p, r) \setminus \{p\}) \cap S \neq \varnothing$.

(b) Any ball $B(p, r)$ contains infinitely many points of S.

(c) There is a sequence $q_1, q_2, q_2, \cdots$ of points of S such that $q_n \neq p$ and $\lim_{n\to\infty} q_n = p$.

Problem 6. Prove that a set $S \subset \mathbf{R}^k$ is bounded if and only if every sequence of points in S has a convergent subsequence.

Problem 7. Prove the following:

(a) The intersection of an arbitrary family of compact sets is compact.

(b) The union of finitely many compact sets is compact.

Problem 8. Give an example of open sets $U_1 \supset U_2 \supset \cdots$ in E such that the intersection $\bigcap_{n=1}^{\infty} U_n$ is closed and nonempty.

Problem 9. Give an example of closed sets $C_1 \supset C_2 \supset \cdots$ in E such that the intersection $\bigcap_{n=1}^{\infty} C_n$ is empty.

Problem 10. Prove or give a counterexample:

(a) The union of infinitely many compact sets is compact.

(b) A non-empty subset S of real numbers which has both a largest and a smallest element is compact.

Problem 11. (a) Consider a sequence of closed intervals $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, $\ldots$. Suppose that $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for all n. Prove that there is a point x which is in every I_n.

(b) Prove that if $\text{Length}\, I_n \to 0$, then the point x in (a) is unique.

(c) Show that this conclusion in (a) is false if we consider open intervals instead of closed intervals. Is it true if we consider open and bounded intervals?