Problem 1. Verify that the following functions are distances on the set $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ of ordered pairs of real numbers.

(a) $d((x_1, x_2), (y_1, y_2)) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$ (this is called the standard metric).

(b) $d_1((x_1, x_2)), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$.

(c) $d_\infty((x_1, x_2)), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

(d) $d_c((x_1, x_2)), (y_1, y_2)) = |x_2| + |y_2| + |x_1 - y_1|$ if $x_1 \neq y_1$, and $= |x_2 - y_2|$ if $x_1 = y_1$.

Problem 2. Prove that the subset of $\mathbf{R}^2$ given by $\{(x_1, x_2) \mid x_1 > x_2\}$ is open in $\mathbf{R}^2$ with the standard metric.

Solution. Let $S = \{(x_1, x_2) \mid x_1 > x_2\}$, and let $p = (a_1, a_2)$ with $a_1 > a_2$ be any point in S. If $r = \dfrac{a_1 - a_2}{\sqrt{2}}$, then $r > 0$ and the ball $B(p, r) \subset S$.

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Problem 3. Prove that the subset of $\mathbf{R}^2$ given by $\{(x_1, x_2) \mid x_1 x_2 = 1, x_1 > 0\}$ is closed in $\mathbf{R}^2$ with the standard metric..

Solution. Let $S = \{(x_1, x_2) \mid x_1 x_2 = 1, x_1 > 0\}$. You may prove that S is closed in a variety of ways: by proving that the complement $\complement S$ is open, or by using some of the theorems that characterize closed sets. For example: a set S is closed if and only if whenever a sequence $p_1, p_2, \cdots$ of points in S converges to a point p, then p is in S.

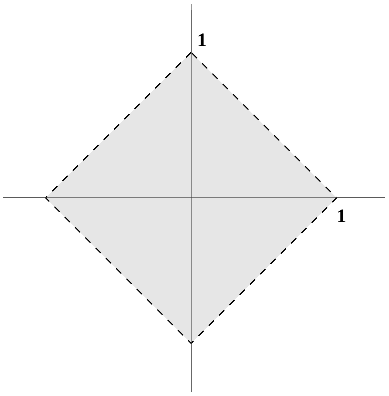
If $p_n = (x_{n,1}, x_{n,2})$ is a sequence in S and p_n converges to $p = (x_1, x_2)$, then the two coordinate sequences satisfy $\lim_{n\to\infty} x_{n,1} = x_1$ and $\lim_{n\to\infty} x_{n,2} = x_2$. Since each p_n is in S, its coordinates satisfy $x_{n,2} = 1/x_{n,1}$ and $x_{n,1} > 0$. Therefore the limit \$ $\lim_{n\to\infty} x_n^1 = x_1 \geq 0$. If $x_1 = 0$, then the sequence $x_{n,2} = 1/x_{n,1}$ will not be bounded, and thus could not be convergent. Thus we have $x_1 > 0$. By the algebra of limits,

$$\begin{aligned} x_2 &= \lim_{n\to\infty} x_{n,2} \\ &= \lim_{n\to\infty} \frac{1}{x_{n,1}} \\ &= \frac{1}{x_1} \end{aligned}$$

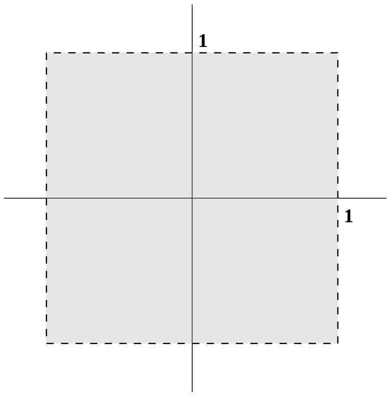
Therefore, the limit $p = (x_1, x_2)$ is in S. ☐

Problem 4. For each of the three distance functions d_1, d_∞, d_c in Problem 1, sketch the open ball with center the origin $(0, 0)$ and radius 1.

Solution. For the distance d_1 and the distance d_c:



For the distance d_∞:



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Problem 5. Prove that the open balls with center $(0, 0)$ and radius 1 for the metrics d_1 and d_∞ in $\mathbf{R}^2$ are open set for the standard metric on $\mathbf{R}^2$.

Problem 6. A point p is an interior point of a set S if p is the center of a ball contained in S, that is, if there is $r > 0$ such that the ball $B(p, r) \subset S$. The set of all interior points of S is called the interior of S, and denoted by $\mathrm{int}(S)$ or S°. Prove that for any set S, the interior S° is the largest open set contained in S. That is:

(a) $S^\circ \subset S$,

(b) S° is an open set, and that

(c) if $U \subset S$ is any open set, then $U \subset S^\circ$.

Solution. (a) If p is in S°, then there is a ball $B(p, r) \subset S$. Hence $p \in S$ because $p \in B(p, r)$.

(b) Let p be a point in S°. We must show that there is a ball with center p that is contained in S°. We know that because $p \in S^\circ$, there is a ball $B(p, r) \subset S$. We claim that in fact, every point in $B(p, r)$ is an interior point of S. Indeed, if q is in $B(p, r)$, then the distance $d(p, q) < r$, and the ball $B(q, r - d(p, q)) \subset B(p, r)$ (this was done in class; use the triangle inequality).

(c) Suppose that $U \subset S$ is open. If p in in U, there is a ball $B(p, r) \subset U$, by the definition of open set. But $B(p, r) \subset U \subset S$ make p be an interior points of S. That is, $U \subset S^\circ$.

□

Problem 7. A point p is a boundary point of a set S if any ball with center p intersects both S and its complement. The boundary of a set S is denoted by ∂S. Find the boundary of the following subsets of $\mathbf{R}$: (a) $\mathbf{R}$; (b) The rational numbers $\mathbf{Q}$; (c) $[0, 1)$; and (d) $\mathbf{N}$.

Solution. (a) Since the complement $\complement\mathbf{R} = \varnothing$, the boundary $\partial\mathbf{R} = \varnothing$; (b) $\partial\mathbf{Q} = \mathbf{R}$; (c) $\partial[0, 1) = \{0, 1\}$; and (d) $\mathbf{N}$. □

Problem 8. The closure of a set S is the union of S and its boundary ∂S. The closure of S is denoted by S^- or $\mathrm{cl}(S)$. Prove that the closure of a set $S \subset \mathbf{R}^n$ is the smallest closed subset of $\mathbf{R}^n$ which contains S.

Solution. By definition, the closure of S is $S^- = S \cup \partial S$, hence S^- contains S.

We show that S^- is closed. Let p be point in of ∂S^-, the boundary of S^-. Then every ball around p intersects S^- and its complement $\complement(S^-)$. Hence there is a sequence (x_n) in S^- such that $|x_n - p| < 1/n$ for all n. Because x_n is in S^-, there is y_n in S such that $|y_n - x_n| < 1/n$. Then, by the triangle inequality, $|y_n - p| \le |y_n - x_n| + |x_n - p| < 2/n$. Thus p is in S^- because the (y_n) is in S and converges to p.

We show that S^- is the smallest closed set containing S, that is, if T is closed and $S \subset T$, then $S^- \subset T$. Assume this was not the case. Then there is p in S^- which is also in T^c. In particular, p is not in S. Moreover, since T^c is open, there is a ball around p which is completely contained in T^c. This ball cannot intersect S because $S \subset T$, and this p cannot be a boundary point of S. Hence p is not in $S \cup \partial S = S^-$, a contradiction. □

Problem 9. Prove that $p_1, p_2, p_3, \cdots$ in a metric space converges to a point p if and only if the sequence of distances $d(p_1, p), d(p_2, p), \cdots$ converges to 0.

Problem 10. Prove that if $\lim p_n = p$, then the set $\{p, p_1, p_2, p_3, \cdots\}$ is closed.

Solution. Let $S = \{p, p_1, p_2, \cdots\}$. We prove that S is closed by proving that its complement is open. If $q \in \complement S$, then $p \neq q$ and the sequence $p_1, p_2, \cdots$ cannot converge to q because sequences can converge to one point at most. By Problem 9, the sequence of distances $d(p, p_n) \to 0$, Since $d(p, q) - d(p_n, p) \le d(p_n, q) \le d(p, q) + d(p_n, p)$ we have that $\lim\limits_{n\to\infty} d(p_n, q) = d(p, q) > 0$. Therefore, there is an N such that $d(p_n, q) > \dfrac{d(p, q)}{2}$ for any $n > N$. Let $r = \min\{d(p_1, q), \cdots, d(p_N, q), d(p, q)/2\}$. Then $r > 0$ because $p_n \neq q$, and the ball $B(q, r) \cap S\varnothing$. □

Problem 11. (a) Find all the cluster (accumulation) points of the set $S = \left\{\dfrac{1}{n} + \dfrac{1}{m} \mid n \text{ and } m \text{ in } \mathbf{N}\right\}$

(b) Prove that p is a cluster point of a set $S \subset \mathbf{R}^n$ if and only if every ball about p contains infinitely many points of S.

Solution. The cluster points of S are $0, 1, 1/2, 1/3, \ldots$. Indeed, each of the numbers $\dfrac{1}{n} = \lim\limits_{n\to\infty} \dfrac{1}{n} + \dfrac{1}{m}$, and $0 = \lim\limits_{n\to\infty} \dfrac{1}{n} + \dfrac{1}{n}$. In each case the point is limit of a sequence whose terms are all in S and are distinct from the point itself.

We prove that those are the only cluster points of S. Suppose that $x \neq 0, 1, 1/2, \cdots$ is an cluster point of S. Then there is a sequence (s_n) in S such that s_n converges to x and $s_n \neq x$ for all n. Each $s_n = 1/p_n + 1/q_n$ for some natural numbers p_n and q_n, with $p_n \le q_n$. If the sequence (q_n) is bounded above, then so is the sequence p_n, and thus there are natural number p and q such that $p_n = q$ and $q_n = q$ for infinitely many n's. This implies that $s_n = 1/p + 1/q$ for infinitely many n's. Since a subsequence of a convergent sequence converges to the same limit, this forces $s_n = 1/p + 1/q = x$ for infinitely many n, which is a contradiction. Thus q_n has a subsequence q_{n_k} which is strictly increasing, hence such that $1/q_{n_k}$ converges to 0. If the corresponding subsequence of natural numbers (p_{n_k}) is bounded above, then it has a constant subsequence $p_{n_{k_l}} = p$ for some natural number p, which implies that $s_{n_{k_l}}$ converges to $1/p$, and thus that $x = 1/p$, again a contradiction. Therefore p_{n_k} has a strictly increasing subsequence $p_{n_{k_l}}$, which implies that $s_{n_{k_l}}$ and thus s_n, converges to 0, also a contradiction. □

Problem 12. (a) Let a_n be a bounded injective sequence of real numbers. Prove that if p is the only cluster (accumulation) point of the set $A = \{a_n \mid n \text{ in } \mathbf{N}\}$, then the sequence a_n converges and $\lim\limits_{n\to\infty} a_n = p$.

(b) Show by a counterexample that this property is not true for unbounded sequences.

Solution. (i) Suppose that a_n does not converge to p. Then there is $\varepsilon > 0$ such that for every natural number k, there is $n_k > k$ such that $|a_{n_k} - p| \ge \varepsilon$. The sequence (a_{n_k}) is a subsequence of (a_n) and thus it is bounded. Therefore it has a subsequence $(a_{n_{k_l}})$ which converges to a point $q \neq p$ (because $|a_{n_{k_l}} - p| > \varepsilon$). Because the sequence (a_n) is injective, all elements of this subsequence are distinct and therefore q is an cluster point of the set A.

Note. If "injective" is not assumed, then (i) may not be true. Let $a_n = 1$ is n is odd and $a_n = 1/n$ if n is even. Then the set $A = \{a_n\} = \{1, 1/2, 1/4, 1/6, \cdots\}$ has only one cluster point, namely 0, but the original sequence a_n does not converge to 0 (or to any other number).

(ii) Let $a_n = n$ if n is odd and $a_n = 1/n$ if n is even. The sequence (a_n) is unbounded and 0 is the only cluster point of the set $A = \{a_n\} = \{1, 1/2, 3, 1/4, 5, 1/6, \cdots\}$. □