Problem 1. Verify that the following functions are distances on the set $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ of ordered pairs of real numbers.

(a) $d((x_1, x_2), (y_1, y_2)) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$ (this is called the standard metric).

(b) $d_1((x_1, x_2)), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$.

(c) $d_\infty((x_1, x_2)), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

(d) $d_c((x_1, x_2)), (y_1, y_2)) = |x_2| + |y_2| + |x_1 - y_1|$ if $x_1 \neq y_1$, and $= |x_2 - y_2|$ if $x_1 = y_1$.

Problem 2. Prove that the subset of $\mathbf{R}^2$ given by $\{(x_1, x_2) \mid x_1 > x_2\}$ is open in $\mathbf{R}^2$ with the standard metric.

Problem 3. Prove that the subset of $\mathbf{R}^2$ given by $\{(x_1, x_2) \mid x_1 x_2 = 1, x_1 > 0\}$ is closed in $\mathbf{R}^2$ with the standard metric..

Problem 4. For each of the three distance functions d_1, d_∞, d_c in Problem 1, sketch the open ball with center the origin $(0, 0)$ and radius 1.

Problem 5. Prove that the open balls with center $(0, 0)$ and radius 1 for the metrics d_1 and d_∞ in $\mathbf{R}^2$ are open set for the standard metric on $\mathbf{R}^2$.

Problem 6. A point p is an interior point of a set S if p is the center of a ball contained in S, that is, if there is $r > 0$ such that the ball $B(p, r) \subset S$. The set of all interior points of S is called the interior of S, and denoted by $\mathrm{int}(S)$ or S°. Prove that for any set S, the interior S° is the largest open set contained in S. That is:

(a) $S^\circ \subset S$,

(b) S° is an open set, and that

(c) if $U \subset S$ is any open set, then $U \subset S^\circ$.

Problem 7. A point p is a boundary point of a set S if any ball with center p intersects both S and its complement. The boundary of a set S is denoted by ∂S. Find the boundary of the following subsets of $\mathbf{R}$.

(a) $\mathbf{R}$

(b) The rational numbers

(c) $[0, 1)$

(d) $\mathbf{N}$

Problem 8. The closure of a set S is the union of S and its boundary ∂S. The closure of S is denoted by S^- or $\mathrm{cl}(S)$. Prove that the closure of a set $S \subset \mathbf{R}^n$ is the smallest closed subset of $\mathbf{R}^n$ which contains S.

Problem 9. Prove that $p_1, p_2, p_3, \cdots$ in a metric space converges to a point p if and only if the sequence of distances $d(p_1, p), d(p_2, p), \cdots$ converges to 0.

Problem 10. Prove that if $\lim p_n = p$, then the set $\{p, p_1, p_2, p_3, \cdots\}$ is closed.

Problem 11. (a) Find all the cluster (accumulation) points of the set $S = \left\{\frac{1}{n} + \frac{1}{m} \mid n \text{ and } m \text{ in } \mathbf{N}\right\}$

(b) Prove that p is a cluster point of a set $S \subset \mathbf{R}^n$ if and only if every ball about p contains infinitely many points of S.

Problem 12. (a) Let a_n be a bounded injective sequence of real numbers. Prove that if p is the only cluster (accumulation) point of the set $A = \{a_n \mid n \text{ in } \mathbf{N}\}$, then the sequence a_n converges and $\lim_{n\to\infty} a_n = p$.

(b) Show by a counterexample that this property is not true for unbounded sequences.