Problem 1. Prove that

(i) $\lim_{n\to\infty} \sqrt[8]{n^2+1} - \sqrt[8]{n^2} = 0.$

(ii) $\lim_{n\to\infty} \sqrt[8]{n^2+1} - \sqrt[4]{n+1} = 0,$

Solution. (i) use algebraic manipulations similar to those utilized in class to show that $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$.

(ii) Use (i) and the identity $\sqrt[8]{n^2+1} - \sqrt[4]{n+1} = \sqrt[8]{n^2+1} - \sqrt[8]{n^2} + \sqrt[4]{n} - \sqrt[4]{n+1}$

□

Problem 2. Prove that if $a, b \ge 0$, then $\lim_{n\to\infty} \sqrt[n]{a^n + b^n} = \max\{a, b\}$.

Solution. Suppose that $0 \le a \le b$. Then $\max\{a, b\} = b$, and

$$b = \sqrt[n]{b^n} \le \sqrt[n]{a^n + b^n} \le \sqrt[n]{b^n + b^n} = \sqrt[n]{2}b.$$

Since $\lim_{n\to\infty} \sqrt[n]{2} = 1$, the limit $\lim_{n\to\infty} \sqrt[n]{a^n + b^n} = b$ by the "squeeze lemma" for limits (Homework 3, Problem 1(ii)). □

Problem 3. Let a_n be the Fibonacci sequence, $a_1 = a_2 = 1$, $a_{n+2} = a_n + a_{n+1}$.

(i) If $r_n = \dfrac{a_{n+1}}{a_n}$, then prove that $r_{n+1} = 1 + \dfrac{1}{r_n}$.

(ii) Prove that $r = \lim_{n\to\infty} r_n$ exists, and $r = 1 + \dfrac{1}{r}$. Conclude that $r = \dfrac{1+\sqrt{5}}{2}$.

Solution. (i) By definition, $a_{n+2} = a_{n+1} + a_n$. If $r_n = \dfrac{a_{n+1}}{a_n}$, then

$$r_{n+1} = \frac{a_{n+2}}{a_{n+1}} = \frac{a_{n+1} + a_n}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}} = 1 + \frac{1}{r_n}.$$

(ii) After looking at the first few terms of the sequence $(r_n) = (1, 2, 3/2, 5/3, 8/5, \ldots)$, we conjecture that the subsequence of (r_n) given by the terms with even indexes, (r_{2n}), is decreasing and that with odd indexes, (r_{2n-1}), is increasing. This is in fact true and can be proved by induction, using (i). To start, we have $r_1 < r_3$ and $r_4 < r_2$. If we know that $r_{2n} < r_{2n-2}$, then $r_{2n-1} = 1 + \dfrac{1}{r_{2n-2}} < 1 + \dfrac{1}{r_{2n}} = r_{2n+1}$, and similarly, if $r_{2n-1} < r_{2n+1}$, then $r_{2n+2} < r_{2n}$.

Furthermore, the sequences (r_{2n}) and (r_{2n-1}) are bounded because (r_n) is bounded on account of the identity proved in (i):

$$1 \le r_{n+1} = 1 + \frac{r_{n-1}}{r_{n-1} + 1} \le 2.$$

Let $r_e = \lim_{n\to\infty} r_{2n}$ and $r_o = \lim_{n\to\infty} r_{2n-1}$. Then, by the properties of limits and the identities

$$r_{2n} = 1 + \frac{1}{r_{2n-1}} \qquad \text{and} \qquad r_{2n+1} = 1 + \frac{1}{r_{2n}}$$

we have

$$r_e = 1 + \frac{1}{r_o} \qquad \text{and} \qquad r_o = 1 + \frac{1}{r_e}$$

or $r_e r_o = r_e + 1 = r_o + 1$. Therefore, the limit $\lim r_n = r$ exists (Why?) and $r = r_e = r_o$ satisfies $r = 1 + \dfrac{1}{r}$, or $r^2 - r - 1 = 0$. Since r is positive, it must equal the positive solution of the quadratic equation $x^2 - x - 1 = 0$, that is, $r = \dfrac{1+\sqrt{5}}{2}$. □

Problem 4. Prove that a set $S \subset \mathbf{R}$ is bounded if and only if every sequence of points in S has a convergent subsequence.

Solution. Recall that a sequence (x_n) is bounded if there is a number M such that $|x_n| \le M$ for all n.

Assume that $S \subset \mathbf{R}$ is bounded. If (x_n) is a sequence in S, then (x_n) is bounded and thus it has a convergent subsequence.

Assume that S is not bounded. Then for any integer n there is an x_n in S such that $|x_n| > n$. Since any convergent sequence is bounded, the sequence (x_n) cannot have a convergent subsequence. □

Problem 5. (i) If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ do not exist, can $\lim_{n\to\infty}[a_n + b_n]$ or $\lim_{n\to\infty} a_n \cdot b_n$ exist?

(ii) If $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty}[a_n + b_n]$ exists, must $\lim_{n\to\infty} b_n$ exist?

(iii) If $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} b_n$ does not exist, can $\lim_{n\to\infty}[a_n + b_n]$ exist?

(iv) If $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} a_n b_n$ exists, does it follow that $\lim_{n\to\infty} b_n$ exists?

Problem 6 (III.9)**.** Let $a_1, a_2, a_3, \cdot$ be a sequence. Prove that $\lim_{n\to\infty} a_n = a$ if and only if the sequence $a_1, a, a_2, a, a_3, a, a_4, \cdots$ is convergent.

Problem 7 (III.11)**.** Let $a_1, a_2, a_3, \cdots$ be a sequence of real numbers.

(a) Prove that if $\lim_{n\to\infty} a_n = a$, then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a.$$

(b) Is the converse true? Prove or give an example.

Solution. A convergent sequence is bounded; thus there is a number $K \ge 0$ such that $|a_n| \le K$ for all n, and thus that $|a_n - a| \le 2K$ for all n.

If $a_n \to a$, then given $\varepsilon > 0$, let N' be such that $|a_n - a| < \varepsilon/2$ if $n > N'$. Let $N = \max\{N', 4KN'/\varepsilon\}$. If $n > N$, then $n > 4KN'/\varepsilon$, so $2KN'/n < \varepsilon/2$, and also $n > N \ge N'$, so

$$\begin{aligned} \left|\frac{a_1 + a_2 + \cdots + a_n}{n} - a\right| &= \left|\frac{a_1 - a}{n} + \cdots + \frac{a_{N'} - a}{n} + \frac{a_{N'+1} - a}{n} + \cdots \right. \\ &\le \frac{|a_1 - a|}{n} + \cdots + \frac{|a_{N'} - a|}{n} + \frac{|a_{N'+1} - a|}{n} + \\ &\le \frac{N'}{n} 2K + \frac{n\varepsilon}{2n} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

□

Problem 8 (III.18). Let $a_1, a_2, a_3, \cdots$ be a bounded sequence of real numbers, and let

$$b_n = \text{l. u. b.}\{a_n, a_{n+1}, a_{n+2}, \cdots\}$$

(a) Prove that the sequence $b_1, b_2, b_3, \cdots$ converges. The limit $\lim_{n\to\infty} b_n$ is denoted by $\limsup_{n\to\infty} a_n$.

(b) Find $\limsup_{n\to\infty} a_n$ for each of the following:

(i) $a_n = \dfrac{1}{n}$

(ii) $a_n = (-1)^n \dfrac{1}{n}$

(iii) $a_n = (-1)^n \dfrac{n}{n+1}$

Proof. (a) If the sequence (a_n) is bounded, then there is $M \geq 0$ such that $|a_n| \leq M$ for all n, and therefore that $|b_n| \leq M$ for all n. Thus the sequence (b_n) is also bounded. Moreover, $b_{n+1} \leq b_n$ for all n because

$$\{a_{n+1}, a_{n+2}, \cdots\} \subset \{a_n, a_{n+1}, a_{n+2}, \cdots\}$$

Thus (b_n) converges because it is a bounded decreasing sequence.

(b) (i) 0; (ii) 0; (iii) 1.

□

Problem 9 (III.18). Let $a_1, a_2, a_3, \cdots$ be a bounded sequence of real numbers.

(a) Define $\liminf_{n\to\infty} a_n$ by

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \text{g. l. b.}\{a_n, a_{n+1}, a_{n+2}, \cdots\}$$

(you must prove that this definition is correct, namely, that the limit exists, as in Problem 8) and prove that

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$$

(b) Prove that $\lim_{n\to\infty} a_n$ exists if and only if $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$, and that in this case, $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$.

Problem 10 (III.19). Let $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$ be bounded sequences of real numbers. Prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

with equality holding inf and only if one of the original sequences converges.