

**Problem 1.** Prove that

$$(i) \lim_{n \rightarrow \infty} \sqrt[8]{n^2 + 1} - \sqrt[8]{n^2} = 0.$$

$$(ii) \lim_{n \rightarrow \infty} \sqrt[8]{n^2 + 1} - \sqrt[4]{n + 1} = 0,$$

**Problem 2.** Prove that if  $a, b \geq 0$ , then  $\lim_{n \rightarrow \infty} \sqrt[n]{a^n + b^n} = \max\{a, b\}$ .

**Problem 3.** Let  $a_n$  be the Fibonacci sequence,  $a_1 = a_2 = 1$ ,  $a_{n+2} = a_n + a_{n+1}$ .

$$(i) \text{ If } r_n = \frac{a_{n+1}}{a_n}, \text{ then prove that } r_{n+1} = 1 + \frac{1}{r_n}.$$

$$(ii) \text{ Prove that } r = \lim_{n \rightarrow \infty} r_n \text{ exists, and } r = 1 + \frac{1}{r}. \text{ Conclude that } r = \frac{1 + \sqrt{5}}{2}.$$

**Problem 4.** Prove that a set  $S \subset \mathbf{R}$  is bounded if and only if every sequence of points in  $S$  has a convergent subsequence.

**Problem 5.** (i) If  $\lim_{n \rightarrow \infty} a_n$  and  $\lim_{n \rightarrow \infty} b_n$  do not exist, can  $\lim_{n \rightarrow \infty} [a_n + b_n]$  or  $\lim_{n \rightarrow \infty} (f \cdot g)(x)$  exist?

$$(ii) \text{ If } \lim_{n \rightarrow \infty} a_n \text{ exists and } \lim_{n \rightarrow \infty} [a_n + b_n] \text{ exists, must } \lim_{n \rightarrow \infty} b_n \text{ exist?}$$

$$(iii) \text{ If } \lim_{n \rightarrow \infty} a_n \text{ exists and } \lim_{n \rightarrow \infty} b_n \text{ does not exist, can } \lim_{n \rightarrow \infty} [a_n + b_n] \text{ exist?}$$

$$(iv) \text{ If } \lim_{n \rightarrow \infty} a_n \text{ exists and } \lim_{n \rightarrow \infty} a_n b_n \text{ exists, does it follow that } \lim_{n \rightarrow \infty} b_n \text{ exists?}$$

**Problem 6 (III.9).** Let  $a_1, a_2, a_3, \dots$  be a sequence. Prove that  $\lim_{n \rightarrow \infty} a_n = a$  if and only if the sequence  $a_1, a, a_2, a, a_3, a, a_4, \dots$  is convergent.

**Problem 7 (III.11).** Let  $a_1, a_2, a_3, \dots$  be a sequence of real numbers.

$$(a) \text{ Prove that if } \lim_{n \rightarrow \infty} a_n = a, \text{ then}$$

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \dots + a_n}{n} = a.$$

$$(b) \text{ Is the converse true? Prove or give an example.}$$

**Problem 8 (III.18).** Let  $a_1, a_2, a_3, \dots$  be a bounded sequence of real numbers, and let

$$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \dots\}$$

$$(a) \text{ Prove that the sequence } b_1, b_2, b_3, \dots \text{ converges. The limit } \lim_{n \rightarrow \infty} b_n \text{ is denoted by } \limsup_{n \rightarrow \infty} a_n.$$

$$(b) \text{ Find } \limsup_{n \rightarrow \infty} a_n \text{ for each of the following:}$$

$$(i) a_n = \frac{1}{n}$$

$$(ii) a_n = (-1)^n \frac{1}{n}$$

$$(iii) a_n = (-1)^n \frac{n}{n+1}$$

**Problem 9 (III.18).** Let  $a_1, a_2, a_3, \dots$  be a bounded sequence of real numbers.

(a) Define  $\liminf_{n \rightarrow \infty} a_n$  by

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \inf\{a_n, a_{n+1}, a_{n+2}, \dots\}$$

(you must prove that this definition is correct, namely, that the limit exists, as in Problem 8) and prove that

$$\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$$

(b) Prove that  $\lim_{n \rightarrow \infty} a_n$  exists if and only if  $\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n$ , and that in this case,  $\lim_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n$ .

**Problem 10** (III.19). Let  $a_1, a_2, a_3, \dots$  and  $b_1, b_2, b_3, \dots$  be bounded sequences of real numbers. Prove that

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$$

with equality holding if and only if one of the original sequences converges.