

Problem 1. (i) Prove that if $a_n \leq b_n$, if $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$, then $a \leq b$.

(ii) Prove that if $a_n \leq c_n \leq b_n$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = l$, then $\lim_{n \rightarrow \infty} c_n = l$.

Solution. (i) Suppose that $a > b$ and apply the definition of limit to $\epsilon = \frac{a-b}{2}$. There is a natural number N_1 such that if $n > N_1$, then $|a_n - a| < \frac{a-b}{2}$ and a natural number N_2 such that if $n > N_2$, then $|b_n - b| < \frac{a-b}{2}$. If $n > \max\{N_1, N_2\}$, then

$$b_n < \frac{a-b}{2} + b = a - \frac{a-b}{2} < a_n,$$

which contradicts the hypothesis. □

Problem 2. Verify the following limits

(i) $\lim_{n \rightarrow \infty} \frac{3n^3 + 7n^2 + 1}{4n^3 - 8n + 63} = \frac{3}{4}$.

(ii) $\lim_{n \rightarrow \infty} \frac{2^n + (-1)^n}{2^{n+1} + (-1)^{n+1}} = \frac{1}{2}$.

Solution. (i) Divide numerator and denominator by n^3 and obtain:

$$\frac{3n^3 + 7n^2 + 1}{4n^3 - 8n + 63} = \frac{3 + \frac{7}{n} + \frac{1}{n^3}}{4 - \frac{8}{n^2} + \frac{63}{n^3}}$$

The limit of the numerator is (by using the algebraic properties of limits):

$$\lim_{n \rightarrow \infty} 3 + \frac{7}{n} + \frac{1}{n^3} = \lim_{n \rightarrow \infty} 3 + 7 \lim_{n \rightarrow \infty} \frac{1}{n} + \lim_{n \rightarrow \infty} \frac{1}{n^3} = 3$$

and by similar arguments, the limit of the denominator is

$$\lim_{n \rightarrow \infty} 4 - \frac{8}{n^2} + \frac{63}{n^3} = 4$$

Since this limit is $\neq 0$, we have

$$\lim_{n \rightarrow \infty} \frac{3n^3 + 7n^2 + 1}{4n^3 - 8n + 63} = \frac{3}{4}$$

(ii) Divide numerator and denominator by 2^{n+1} . □

Problem 3. Prove that if $a > 0$, then $\lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$.

Problem 4. Prove that $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$. (Hint: put $\sqrt[n]{n} = 1 + a_n$, prove that $a_n > 0$ for $n > 1$, deduce that $n - 1 \geq \frac{1}{2}n(n-1)a_n^2$ for $n > 1$, hence that $0 \leq a_n^2 \leq 2/n$.)

Problem 5. Does the sequence a_1, a_2, a_3, \dots converge or diverge? If it converges, what is the limit?

(i) $a_n = \frac{n}{n+1} - \frac{n+1}{n}$.

(ii) $a_n = \frac{2^n}{n!}$.

(iii) $a_n =$ the n th decimal digit of $\sqrt{2}$ (thus $a_1 = 4, a_2 = 1, a_3 = 4, a_4 = 2$, and so on).

Solution. (i) 0, (ii) 0, (iii) Does not converge. □

Problem 6. Prove or give a counterexample. Let a_1, a_2, \dots be a sequence such that $\lim_{n \rightarrow \infty} (a_{n+1} - a_n) = 0$. Does a_n have to converge?

Solution. Cf. Problem 8 below. □

Problem 7. Prove that the sequence x_1, x_2, x_3, \dots of real numbers defined by $x_1 = 1$ and $x_{n+1} = x_n + \frac{1}{x_n^2}$ is unbounded.

Solution. It follows that $1 = x_1 \leq x_2 \leq x_3 \leq \dots$. Suppose that the sequence (x_n) is bounded. Then there is $\lim_{n \rightarrow \infty} x_n = x$. Using the formula $x_{n+1} = x_n + 1/x_n^2$ and the algebraic properties of limits, the limit x must then satisfy $x = x + \frac{1}{x^2}$, or $\frac{1}{x^2} = 0$, which is impossible. □

Problem 8. Prove or give a counterexample:

(i) If (a_n) is an increasing sequence (that is, $a_1 \leq a_2 \leq a_3 \leq \dots$) such that $\lim_{n \rightarrow \infty} (a_{n+1} - a_n) = 0$, then (a_n) is convergent.

(ii) If (a_n) is increasing and bounded above, and $\lim_{n \rightarrow \infty} a_n = a$, then $a_n \leq a$.

Solution. (i) Let $a_n = \sqrt{n}$. Then $a_n < a_{n+1}$ and (rationalizing)

$$0 < a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{(n+1)n}} \leq \frac{1}{n+1}$$

which implies that $\lim_{n \rightarrow \infty} (a_{n+1} - a_n) = 0$.

(ii) If there is a natural number N such that $a_N > a$, then $a_n \geq a_N > a$ for all natural numbers $n \geq N$, and a_n cannot converge to a . □

Problem 9. (i) Give an example of a sequence of real numbers with subsequences converging to every integer.

(ii) Give an example of a sequence of real numbers with subsequences converging to every real number.

Solution. (i) 0, -1, 0, 1, -2, -1, 0, 1, 2, -3, -2, -1, 0, 1, 2, 3, ... and so on. Each integer appear infinitely many times in this sequence and thus you can extract a subsequence which converges to any integer (in fact, a constant sequence). □

Problem 10. Prove that if the subsequences (a_{2n}) and (a_{2n+1}) of a sequence (a_n) of real numbers both converge to the same limit l , then (a_n) converges to l .

Solution. Given $\varepsilon > 0$ there are natural numbers N_e and N_o such that if n is an even natural number and $n > N_e$, then $|a_n - 1| < \varepsilon$, and if n is an odd natural number and $n > N_o$, then $|a_n - 1| < \varepsilon$. Let $N = \max\{N_e, N_o\}$. If n is a natural number and $n > N$, then n is either even and $> N_e$, or n is odd and $> N_o$. In either case, $|a_n - 1| < \varepsilon$. \square

Problem 11. Let (a_n) be a sequence of real numbers such that $0 < a_1 < 1$, and $a_{n+1} = \frac{2}{1 + a_n}$ for all $n > 1$.

- (i) Prove that the subsequences (a_{2n}) and (a_{2n+1}) are both monotonic, one decreasing and the other increasing.
- (ii) Prove that (a_n) converges, and find its limit.

Solution. (ii) From the definition of $a_{n+1} = \frac{2}{1 + a_n}$ we obtain that

$$a_{2n} = \frac{2}{1 + a_{2n-1}} \quad \text{and} \quad a_{2n+1} = \frac{2}{1 + a_{2n}}.$$

Therefore if the even indexed subsequence has $\lim_{n \rightarrow \infty} a_{2n} = x$ and the odd indexed subsequence has $\lim_{n \rightarrow \infty} a_{2n} = y$, we obtain that

$$x = \frac{2}{1 + y} \quad \text{and} \quad y = \frac{2}{1 + x}$$

These two equations imply that $x = y$, and therefore that $\lim_{n \rightarrow \infty} a_n = x = y$. Furthermore, x satisfies $x = \frac{2}{1 + x}$, or $x^2 + x - 2 = 0$, or $x = -2, 1$. Since $a_n \geq 0$, $x = \lim_{n \rightarrow \infty} a_n \geq 0$, and so it must be that $x = 1$. \square