Problem 1. (i) Prove that if $a_n \leq b_n$, if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $a \leq b$.

(ii) Prove that if $a_n \leq c_n \leq b_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l$, then $\lim_{n\to\infty} c_n = l$.

Problem 2. Verify the following limits

(i) $\lim_{n\to\infty} \frac{3n^3 + 7n^2 + 1}{4n^3 - 8n + 63} = \frac{3}{4}$.

(ii) $\lim_{n\to\infty} \frac{2^n + (-1)^n}{2^{n+1} + (-1)^{n+1}} = \frac{1}{2}$.

Problem 3. Prove that if $a > 0$, then $\lim_{n\to\infty} \sqrt[n]{a} = 1$.

Problem 4. Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. (Hint: put $\sqrt[n]{n} = 1 + a_n$, prove that $a_n > 0$ for $n > 1$, deduce that $n - 1 \geq \frac{1}{2}n(n-1)a_n^2$ for $n > 1$, hence that $0 \leq a_n^2 \leq 2/n$.)

Problem 5. Does the sequence $a_1, a_2, a_3, \cdots$ converge or diverge? If it converges, what is the limit?

(i) $a_n = \frac{n}{n+1} - \frac{n+1}{n}$.

(ii) $a_n = \frac{2^n}{n!}$.

(iii) $a_n =$ the nth decimal digit of $\sqrt{2}$ (thus $a_1 = 4$, $a_2 = 1$, $a_3 = 4$, $a_4 = 2$, and so on).

Problem 6. Prove or give a counterexample. Let $a_1, a_2, \cdots$ be a sequence such that $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$. Does a_n have to converge?

Problem 7. Prove that the sequence $x_1, x_2, x_2, \cdots$ of real numbers defined by $x_1 = 1$ and $x_{n+1} = x_n + \frac{1}{x_n^2}$ is unbounded.

Problem 8. Prove or give a counterexample:

(i) If (a_n) is an increasing sequence (that is, $a_1 \leq a_2 \leq a_3 \leq \cdots$) such that $\lim_{n\to\infty} \big(a_{n+1} - a_n\big) = 0$, then (a_n) is convergent.

(ii) If (a_n) is increasing and bounded above, and $\lim_{n\to\infty} a_n = a$, then $a_n \leq a$.

Problem 9. (i) Give an example of a sequence of real numbers with subsequences converging to every integer.

(ii) Give an example of a sequence of real numbers with subsequences converging to every real number.

Problem 10. Prove that if the subsequences (a_{2n}) and (a_{2n+1}) of a sequence (a_n) of real numbers both converge to the same limit l, then (a_n) converges to l.

Problem 11. Let (a_n) be a sequence of real numbers such that $0 < a_1 < 1$, and $a_{n+1} = \frac{2}{1 + a_n}$ for all $n > 1$.

(i) Prove that the subsequences (a_{2n}) and (a_{2n+1}) are both monotonic, one decreasing and the other increasing.

(ii) Prove that (a_n) converges, and find its limit.