**Problem 1.** Prove that if $0 < a < b$, then

$$a < \sqrt{ab} < \frac{a+b}{2} < b.$$

Notice that the inequality $\sqrt{ab} \le (a+b)/2$ holds for all $a, b \ge 0$.

**Solution.** If $a < b$, then

$$a = \frac{a+a}{2} < \frac{a+b}{2} < \frac{b+b}{2} = b.$$

If $0 < a < b$, then $b - a$ is positive and $a$ is positive, so the product $a(b-a)$ is also positive. Thus $ab - a^2 > 0$, or $a^2 < ab$.

Now we want to show that this implies $a < \sqrt{ab}$. Indeed, if $a < \sqrt{ab}$ where false, then either $a = \sqrt{ab}$ or $a > \sqrt{ab}$. But if $a = \sqrt{ab}$, then $a^2 = ab$ and so $a = b$ because $a > 0$. If $a > \sqrt{ab}$ then $aa > a\sqrt{ab}$ and $a\sqrt{ab} > \sqrt{ab}\sqrt{ab}$, so that $a^2 > ab$ and $a > b$ (because $b > 0$).

Moreover, $(b-a)^2 > 0$, so

$$\begin{aligned} a^2 + b^2 &> 2ab, \\ a^2 + b^2 + 2ab &> 4ab, \\ (a+b)^2 &> 4ab, \end{aligned}$$

so $(a+b) > 2\sqrt{ab}$ (by a reasoning similar to that in the previous paragraph). Moreover, for all $a, b$, $(a-b)^2 \ge 0$, and thus $(a+b)^2 \ge 4ab$, which implies that $a + b \ge 2\sqrt{ab}$ for $a, b \ge 0$.

**Remark.** For convenience, I will isolate three properties used in the solution of the previous exercise.

(i) If $a < b$ and $c > 0$, then $ac < bc$.

Indeed, $b - a$ is positive and $c$ is also positive, hence the product $(b-a)c = bc - ac$ is positive, that is $ac < bc$.

(ii) If $0 \le a < b$, then $a^2 < b^2$.

Indeed, by (i), $aa < ab$ and $ba < bb$, so that $a^2 < ab < b^2$.

(iii) If $a, b \ge 0$ and $a^2 < b^2$, then $a < b$.

Indeed, if $a < b$ were false, then either $a = b$ or $a > b$. If $a = b$, then $a^2 = b^2$, and if $a > b \ge 0$, then $a^2 > b^2$, by (ii).

**Problem 2.** In Problem II.7 you proved that

$$\max\{a, b\} = \frac{a + b + |b-a|}{2}.$$

Derive a similar formula for $\max\{a, b, c\}$, using, for example, $\max\{a, b, c\} = \max\{a, \max\{b, c\}\}$.

*Solution.*

$$\begin{aligned} \max(x,y,z) &= \max(x, \max(y,z)) \\ &= \frac{x + \max(y,z) + |\max(y,z) - x|}{2} \\ &= \frac{x + \dfrac{y+z+|z-y|}{2} + \left|\dfrac{y+z+|z-y|}{2} - x\right|}{2} \\ &= \frac{|y-z| + y + z + 2x + |y+z+|y-z| - 2x|}{4} \end{aligned}$$

□

**Problem 3.** Prove that if

$$|x - x_0| < \frac{\varepsilon}{2} \quad \text{and} \quad |y - y_0| < \frac{\varepsilon}{2},$$

then

$$\big|(x+y) - (x_0+y_0)\big| < \varepsilon,$$

and

$$\big|(x-y) - (x_0-y_0)\big| < \varepsilon.$$

*Solution.*

$$\begin{aligned} \big|(x+y) - (x_0+y_0)\big| &= \big|(x-x_0) + (y-y_0)\big| \\ &\le |x-x_0| + |y-y_0| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

and

$$\begin{aligned} \big|(x-y) - (x_0-y_0)\big| &= \big|(x-x_0) + (y_0-y)\big| \\ &\le |x-x_0| + |y_0-y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

□

**Problem 4.** Prove that if

$$|x-x_0| < \frac{\varepsilon}{2(|y_0|+1)} \quad \text{and} \quad |x-x_0| < 1 \quad \text{and} \quad |y-y_0| < \frac{\varepsilon}{2(|x_0|}$$

then

$$\big|xy - x_0y_0\big| < \varepsilon\ .$$

(Hint. Write $xy - x_0y_0$ in a way that involves $x - x_0$ and $y - y_0$.)

**Problem 5.** Prove that if $x_0 \ne 0$ and

$$|x - x_0| < \min\left\{\frac{|x_0|}{2}, \frac{\varepsilon|x_0|^2}{2}\right\},$$

then $x \ne 0$ and

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| < \varepsilon.$$

**Problem 6.** Find the greatest lower bound and the least upper bound, if they exist, of the set $\{x \in \mathbf{R} \mid x < 0 \text{ and } x^2 + x - 1 < 0\}$. Does this set have a maximum? a minimum?

**Problem 7.** If $A \ne \emptyset$ is bounded below, let $B$ be the set of all lower bounds of $A$. Prove that: (1) $B \ne \emptyset$, (2) $B$ is bounded above, and (3) $\text{l.u.b.}\, B = \text{g.l.b.}\, A$.

*Solution.* The set $B \neq \emptyset$ because $A$ is bounded below (any lower bound for $A$ is in $B$). Because $A$ is nonempty, there is $a$ in $A$, and this $a$ satisfies $y \leq a$ for all $y$ in $B$. Because of this, $y \leq x$ for all $x$ in $A$ and all $y$ in $B$, and thus $\text{l. u. b.}\, B \leq \text{g. l. b.}\, A$.

If $\text{l. u. b.}\, B < \text{g. l. b.}\, A$, then there is a number $x$ such that $\text{l. u. b.}\, B < x < \text{g. l. b.}\, A$. The inequality $\text{l. u. b.}\, B < x$ implies that $b < x$ for all $b$ in $B$ and thus that $x$ is not in $B$. The inequality $x < \text{g. l. b.}\, A$ implies that $x < a$ for all $a$ in $A$, and thus that $x$ is a lower bound for $A$. Therefore $x$ is in $B$. But this contradicts the inequality $\text{l. u. b.}\, B < x$ as noted. □

**Problem 8.** Let $a$ be a real number. Prove that for any real number $\varepsilon > 0$ there is a natural number $n$ such that $\dfrac{a}{2^n} < \varepsilon$

**Problem 9.** Let $A$ be a nonempty set of real numbers. Prove that $\alpha = \text{l. u. b.}\, A$ if and only if $\alpha$ is an upper bound for $A$ and for any $\varepsilon > 0$ there is $x$ in $A$ such that $\alpha < x + \varepsilon$.

**Problem 10.** Let $A$ and $B$ be two nonempty sets of real numbers which are bounded above, and let $A+B$ denote the set of all real numbers of the form $x+y$ with $x$ in $A$ and $y$ in $B$. Prove that $\text{l. u. b.}(A+B) = \text{l. u. b.}\, A + \text{l. u. b.}\, B$.

*Solution.* The inequality $\text{l. u. b.}(A+B) \leq \text{l. u. b.}\, A + \text{l. u. b.}\, B$ should be easy: if $z$ is in $A+B$, then $z = x+y$ for some $x$ in $A$ and some $y$ in $B$, hence $z = x+y \leq \text{l. u. b.}\, A + \text{l. u. b.}\, A$ because $x \leq \text{l. u. b.}\, A$ and $y \leq \text{l. u. b.}\, B$ and thus $\text{l. u. b.}(A+B) \leq \text{l. u. b.}\, A + \text{l. u. b.}\, B$.

To prove that $\text{l. u. b.}\, A + \text{l. u. b.}\, B \leq \text{l. u. b.}(A+B)$, it suffices to prove that $\text{l. u. b.}\, A + \text{l. u. b.}\, B \leq \text{l. u. b.}(A+B) + \varepsilon$ for all $\varepsilon > 0$. For this, begin by choosing $x$ in $A$ and $y$ in $B$ with $\text{l. u. b.}\, A - x < \varepsilon/2$ and $\text{l. u. b.}\, B - y < \varepsilon/2$ (use Problem 9). Then $\text{l. u. b.}\, A + \text{l. u. b.}\, B - (x+y) < \varepsilon$, where $x+y \in A+B$. □