Problem 1. Prove that if $0 < a < b$, then

$$a < \sqrt{ab} < \frac{a+b}{2} < b.$$

Notice that the inequality $\sqrt{ab} \le (a+b)/2$ holds for all $a, b \ge 0$.

Problem 2. In Problem II.7 you proved that

$$\max\{a, b\} = \frac{a + b + |b - a|}{2}.$$

Derive a similar formula for $\max\{a, b, c\}$, using, for example, $\max\{a, b, c\} = \max\{a, \max\{b, c\}\}$.

Problem 3. Prove that if

$$|x - x_0| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |y - y_0| < \frac{\varepsilon}{2},$$

then

$$\big|(x + y) - (x_0 + y_0)\big| < \varepsilon,$$

and

$$\big|(x - y) - (x_0 - y_0)\big| < \varepsilon.$$

Problem 4. Prove that if

$$|x - x_0| < \frac{\varepsilon}{2(|y_0| + 1)} \quad \text{and} \quad |x - x_0| < 1 \quad \text{and} \quad |y - y_0| < \frac{\varepsilon}{2(|x_0| + 1)},$$

then

$$\big|xy - x_0 y_0\big| < \varepsilon\ .$$

(Hint. Write $xy - x_0y_0$ in a way that involves $x - x_0$ and $y - y_0$.)

Problem 5. Prove that if $x_0 \neq 0$ and

$$|x - x_0| < \min\left\{\frac{|x_0|}{2}, \frac{\varepsilon|x_0|^2}{2}\right\},$$

then $x \neq 0$ and

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| < \varepsilon.$$

Problem 6. Find the greatest lower bound and the least upper bound, if they exist, of the set $\{x \in \mathbf{R} \mid x < 0 \text{ and } x^2 + x - 1 < 0\}$. Does this set have a maximum? a minimum?

Problem 7. If $A \neq \emptyset$ is bounded below, let B be the set of all lower bounds of A. Prove that: (1) $B \neq \emptyset$, (2) B is bounded above, and (3) $\text{l.u.b.}\, B = \text{g.l.b.}\, A$.

Problem 8. Let a be a real number. Prove that for any real number $\varepsilon > 0$ there is a natural number n such that $\dfrac{a}{2^n} < \varepsilon$

Problem 9. Let A be a nonempty set of real numbers. Prove that $\alpha = \text{l.u.b.}\, A$ if and only if α is an upper bound for A and for any $\varepsilon > 0$ there is x in A such that $\alpha < x + \varepsilon$.

Problem 10. Let A and B be two nonempty sets of real numbers which are bounded above, and let $A + B$ denote the set of all real numbers of the form $x + y$ with x in A and y in B. Prove that $\text{l.u.b.}(A + B) = \text{l.u.b.}\, A + \text{l.u.b.}\, B$.