**Problem 1** (II.3)**.** Prove that if $a < b < 0$, then $1/a > 1/b$.

*Proof.* If $a < b < 0$, then $ab > 0$ and so $1/(ab) > 0$. Therefore, $1/b = a(1/ab) < b(1/ab) = 1/a$. □

**Problem 2.** For which real numbers $x$ are the following inequalities true?

(c) $\frac{2}{x} > x - 1$.

*Proof.* (c) First do some algebra of inequalities:

$$\begin{aligned} x - 1 &< \frac{2}{x} \\ x - 1 - \frac{2}{x} &< 0 \quad \text{(subtr. } 2/x\text{)} \\ \frac{x^2 - x - 2}{x} &< 0 \quad \text{(common denom.)} \end{aligned}$$

Recall that a product $a \cdot b < 0$ if and only if $(a < 0 \wedge b > 0) \vee (a > 0 \wedge b < 0)$. Therefore, $\frac{x^2 - x - 2}{x} < 0$ if and only if $(x^2 - x - 2 < 0 \wedge 1/x > 0) \vee (x^2 - x - 2 > 0 \wedge 1/x < 0)$. It is elementary, by solving a quadratic equation, to show that $x^2 - x - 2 < 0$ if and only if $-1 < x < 2$. Since $1/x$ and $x$ are both $> 0$ or both $< 0$, we immediatly obtain that the inequality (c) holds if and only if $0 < x < 2$ or $x < -1$. Using interval notation, the set of $x$ where (c) holds is $(-\infty, -1) \cup (0, 2)$. □

**Problem 3** (II.7)**.** For any $a, b$,

$$\max\{a, b\} = \frac{a + b + |a - b|}{2}$$

*Proof.* If $a \le b$, then $|b - a| = b - a$, so $a + b + |b - a| = a + b + b - a = 2b$ which is $2\max\{a, b\}$. Interchanging $a$ and $b$ proves the formula when $a \ge b$. The same type of argument works for $\min\{a, b\}$. □

**Problem 4** (II.10)**.** Find the g. l. b. and l. u. b. of the following sets, if they exist.

(a) $A = \{1, 1/2, 1/3, \cdots\}$

(b) $B = \{1/3, 4/9, 13/27, \cdots\}$

*Proof.* (a) The least upper bound l. u. b. $A = 1$ because $1 \ge 1/n$ for any positive integer $n$ and $1 \in A$.

The greatest lower bound g. l. b. $A = 0$. Indeed, $0$ is a lower bound for $A$, and if $x > 0$ were another lower bound, then $x \le 1/n$ for every positive integer $n$, or $n \le 1/x$ for every $n$, implying that the set of positive integers is bounded above.

(b) In the order in which they are written, two consecutive numbers $p/q, r/s$ in $B$ are related by $r = p + q$ and $s = 3q$. That is, if $b$ is in $B$, then $(1/3)(b + 1)$ is also in $B$.

This immediatly implies that $1/3 =$ g. l. b. $B$ because $1/3 \in B$ and if $b \ge 1/3$, then $b/3 + 1/3 \ge 1/3$. (Write the details.)

The l. u. b. $B = 1/2$. Indeed, if $b$ in $B$ satisfies $b \le 1/2$, then the next number in the list satisfies $b/3 + 1/3 \le 1/6 + 1/3 = 1/2$, so $1/2$ is an upper bound for $B$. If $x$ is any upper bound for $B$, then $b \le x$ for all $b$ in $B$ and so $b/3 + 1/3 \le x$ for all $b$ in $B$ (because $b \in B \Rightarrow b/3 + 1/3 \in B$). Now $b/3 + 1/3 \le x$ implies $b \le 3x - 1$, that is, if $x$ is an upper bound for $B$, the $3x - 1$ is an upper bound for $B$. In particular, if $x =$ l. u. b. $B$, then $x \le 3x - 1$, or $1/2 \le x$. □

**Problem 5** (II.11)**.** Prove that if $a > 1$ is a real number, that the set $\{a, a^2, a^3, \cdots\}$ is not bounded above.

*Proof.* To show that if $a > 1 + 1/n$ then $a^n > (1 + 1/n)^n > 2$, apply the Binomial theorem to $(1 + 1/n)^n = 1 + n(1/n) + n(n-1)1/n^2 + \cdots + 1/n^n$. □

**Problem 6** (II.12)**.** Prove that if $X, Y$ are nonempty subsets of $\mathbf{R}$ whose union is $\mathbf{R}$, and $x \le y$ for any $x$ in $X$ and any $y$ in $Y$, then there is $\alpha$ such that $X$ is one of the sets $\{x \le \alpha\}$ or $\{x < \alpha\}$.

*Proof.* Note that the set $X \cap Y$ has at most one point: if $a, b \in X \cap Y$, then both $a \le b$ and $b \le a$, so $a = b$.

Because $Y \neq \varnothing$, any number in $Y$ is an upper bound for $X$. Since $X \neq \varnothing$, there is $\alpha =$ l. u. b. $X$. If $a < \alpha$, then $a \in X$, for otherwise $a \in Y$ (because $X \cup Y = \mathbf{R}$) and then $a$ would be an upperbound for $X$ smaller than $\alpha$. Therefore, $\{a \in \mathbf{R} \mid a < \alpha\} \subset X$. Similarly, if $b > \alpha$, then $b \in Y$ for if not $b \in X$ and that $\alpha$ could not be an upper bound for $X$. Therefore, $X \subset \{a \in \mathbf{R} \mid a \le \alpha\}$. □