

Problem 1 (I.2). If A is a subset of S , prove that:

(a) $\mathcal{C}(\mathcal{C}A) = A$

(d) $A \times \emptyset = \emptyset \times A = \emptyset$.

Solution. (a) Recall that if $A \subset S$, then $\mathcal{C}A$ denotes the complement of A , defined by $\mathcal{C}A = \{x \in S \mid x \notin A\}$. It is also denoted by $S \setminus A$. Thus, x is in $\mathcal{C}(\mathcal{C}A)$ if and only if x is not in $\mathcal{C}A$ if and only if x is in A .

(d) The Cartesian product of two sets $A \times \emptyset = \{(x, y) \mid x \in A, y \in \emptyset\}$. Since there is no y in \emptyset , there is no (x, y) in $A \times \emptyset$. \square

Problem 2 (I.4). If A, B, C are sets, prove that:

(b) $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$

(c) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap B \cap C)$

Solution. (b) (with words) x is in $(A \cup B) \setminus (A \cap B)$ if and only if: (1) x is in $A \cup B$ and (2) x is not in $A \cap B$. That is, if and only if: (1') x is in A or B but (2') x is not in both A and B . Combining the alternatives in (1') and (2'), this is equivalent to saying that either (1'') x is in A and not in B , or (2'') x is in B and not in A , that is, x is in $(A \setminus B) \cup (B \setminus A)$.

(c) (With symbols) The symbol \vee stand for nonexclusive "or" and the symbol \wedge stands for "and." The key observation is that $x \notin B \setminus C$ if and only if $(x \notin B \wedge x \in C) \vee (x \notin B \wedge x \notin C)$.

$$\begin{aligned} x \in A \setminus (B \setminus C) &\Leftrightarrow \\ &\Leftrightarrow (x \in A) \wedge (x \notin B \setminus C) \\ &\Leftrightarrow (x \in A) \wedge ((x \notin B \wedge x \in C) \vee (x \notin B \wedge x \notin C)) \\ &\Leftrightarrow (x \in A \wedge x \notin B) \vee (x \in A \wedge x \in B \wedge x \in C) \\ &\Leftrightarrow x \in (A \setminus B) \cup (A \cap B \cap C) \end{aligned}$$

\square

Problem 3 (I.5). Let I be a nonempty set and for each i in I , let X_i Prove that:

(a) for any set B we have

$$B \cap \left(\bigcup_{i \in I} X_i \right) = \bigcup_{i \in I} (B \cap X_i)$$

(b) if each X_i is a subset of a given S , then

$$\mathcal{C} \left(\bigcup_{i \in I} X_i \right) = \bigcap_{i \in I} \mathcal{C}X_i.$$

Solution. (a) (with words) If x is in $B \cap (\bigcup_{i \in I} X_i)$, then: (1) x is in B , and (2) x is in $\bigcup_{i \in I} X_i$. (2) is equivalent to (2') there is i_x (some index in I that depends on x) in I such that x is in X_{i_x} . Then (1) and (2') say that x is in B and in X_j ,

which is the same as to say that x is in $\bigcup_{i \in I} (B \cap X_i)$. (b) (with symbols)

$$\begin{aligned} x \in \mathcal{C} \left(\bigcup_{i \in I} X_i \right) &\Leftrightarrow x \notin \bigcup_{i \in I} X_i \\ &\Leftrightarrow x \notin X_i, \forall i \in I \\ &\Leftrightarrow x \in \mathcal{C}X_i, \forall i \in I \\ &\Leftrightarrow x \in \bigcap_{i \in I} \mathcal{C}X_i \end{aligned}$$

\square

Problem 4 (I.8). Prove that $f : X \rightarrow Y$ is one-one if and only if $f^{-1}(f(A)) = A$ for any $A \subset X$.

Solution. Suppose that f is one-one. We must prove that if $A \subset X$, then $f^{-1}(f(A)) \subset A$, because $f^{-1}(f(A)) \supset A$ always holds true by Problem I.7(e). If x is in $f^{-1}(f(A))$, then $f(x)$ is in $f(A)$. Write $y = f(x)$. Then y is in $f(A)$, which means that there is x' in A such that $f(x') = y$. We have then two points x and x' such that $f(x) = y = f(x')$. Because f is one-one, it must be that $x = x'$, and so x is in A .

Suppose that $A = f^{-1}(f(A))$ for all $A \subset X$. Let $x \neq x'$ be such that $f(x) = f(x')$. If $A = \{x\}$, then $f(A) = \{f(x)\} = \{f(x')\}$, and thus both x and x' are in $f^{-1}(f(A)) = f^{-1}(\{f(x)\})$. \square

Problem 5 (I.9). (a) How many subsets are there in the set $\{1, 2, 3, \dots, n\}$? (b) How many maps of this set into itself? (c) How many maps of this set onto itself?

Solution. (a) We can do this by counting: there is 1 subset with 0 elements, namely, \emptyset . There are n subsets with 1 element, namely, $\{1\}, \{2\}, \dots, \{n\}$, and so on. In general, for $0 \leq k \leq n$, there are exactly $\binom{n}{k}$ (n choose k) subsets with exactly k elements. The total number of subsets is thus

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{k} + \dots + \binom{n}{n},$$

and this sum equals $(1 + 1)^n = 2^n$, by the Binomial theorem.

(b) A map from $\{1, \dots, n\}$ into itself corresponds to choosing n elements with replacement from $\{1, \dots, n\}$. There are exactly n^n ways of doing that.

(c) There are $n!$ onto maps. \square

Problem 6 (I.10). (a) How many functions are there from a nonempty set S into \emptyset ?

(b) How many functions are there from \emptyset into an arbitrary set?

Solution. A function f from X into Y is a subset f of $X \times Y$ with the property that x in X appears once and only once as the first element of an ordered pair (x, y) in the subset f .

(a) None. If $S \neq \emptyset$, then there is x in S but there is no y in \emptyset such that (x, y) is in $S \times \emptyset = \emptyset$.

(b) One. For any set S , $\emptyset \times S = \emptyset$. The function (called empty function) $\emptyset \rightarrow S$ corresponds to the empty set $\emptyset = \emptyset \times S$. \square