

12 The Exponential and the Logarithm

Definition 12.1. For $x > 0$, define the **logarithm** of x by $\log x = \int_1^x \iota$, where ι is the function given by $\iota(t) = 1/t$ for $t > 0$.

Theorem 12.1. The function \log is defined for all $x > 0$, is differentiable with derivative $\log' = \iota$, and is therefore strictly increasing on $(0, \infty)$. Furthermore, $\log(xy) = \log x + \log y$, for any $x, y > 0$, and its range is the set of all real numbers.

Proof. The properties that \log is differentiable with $\log' = \iota$ follows from the Fundamental theorem of calculus.

Fix $y > 0$ and set $f(x) = \log(xy)$. Then f is differentiable because it is the composite of two differentiable functions: \log and multiplication by y . By the Chain Rule, $f'(x) = \log'(xy) \cdot y = 1/x$. Thus $f' = \log'$ on the interval $(0, \infty)$, which means that $f = \log + c$ for some number c , by the Intermediate Value Theorem. To find c , set $x = 1$ and get $f(1) = \log(1y) = c$. Therefore $\log(xy) = \log x + \log y$.

It follows that $\log(x^n) = n \log(x)$ for any integer x . Note that $\log 2 > 0$ because $2 > 1$. Hence, if y is any real number, then there are integers n, m such that $n \log 2 < y < m \log 2$. Since \log is continuous, the Intermediate Value Theorem implies that there is a number x between 2^n and 2^m such that $\log x = y$. \square

Definition 12.2. The **exponential function** $\exp : \mathbf{R} \rightarrow (0, \infty)$ is defined by setting $\exp(x) = y$ if and only if $\log y = x$. This makes sense because \log is one-one (strictly increasing) and takes on all real values.

Theorem 12.2. The function \exp is strictly increasing, takes on all positive numbers, and satisfies $\exp(x + y) = \exp(x) \cdot \exp(y)$ for all numbers x, y . Moreover, \exp is differentiable with derivative $\exp' = \exp$.

Proof. The properties of \exp stated in the first sentence follow immediately from its definition as the inverse of \log , and the corresponding properties of the function \log .

To show that \exp is continuous at x_0 , let $\varepsilon < \exp(x_0)$ be given. If $\delta = \min\{x_0 - \log(\exp(x_0) - \varepsilon), \log(\exp(x_0) + \varepsilon) - x_0\}$, and $-\delta < x - x_0 < \delta$, then $\log(\exp(x_0) - \varepsilon) < x < \log(\exp(x_0) + \varepsilon)$, and so $\exp(x_0) - \varepsilon < \exp(x) < \exp(x_0) + \varepsilon$, because \log and \exp is strictly increasing.

If \exp is differentiable at 0, then it is differentiable at any x and $\exp'(x) = \exp'(0) \cdot \exp(x)$, because $\frac{\exp(x+h) - \exp(x)}{h} = \exp(x) \frac{\exp(h) - \exp(0)}{h}$.

To show that \exp is differentiable at 0 and $\exp'(0) = 1$, use that \log is differentiable at $1 = \exp(0)$ and $\log'(1) = 1$ to obtain a function α satisfying $\lim_{k \rightarrow 1} \alpha(k) = 0$ and such that $\log(k) = (k - 1) + \alpha(k)$ for all $k \neq 1$. Then $(\exp(h) - 1) = \log(\exp(h)) - \alpha(\exp(h)) = h - \beta(h)$ where $\beta = \alpha \circ \exp$ satisfies $\lim_{h \rightarrow 0} \beta(h) = 0$ because $\lim_{h \rightarrow 0} \exp(h) = 1$ (by continuity of \exp), which implies that $\exp'(0) = 1$. \square

Theorem 12.3. If $E : \mathbf{R} \rightarrow (0, \infty)$ is a continuous homomorphism from the additive group of all real numbers to the multiplicative group of all positive real numbers, then there is λ such that $E(x) = \exp(\lambda \cdot x)$ for every x .

Proof. Suppose that $E : \mathbf{R} \rightarrow (0, \infty)$ is a continuous function that satisfies $E(x + y) = E(x)E(y)$ and $E(0) = 1$. Then the composite $f = \log \circ E$ is continuous and satisfies $f(x + y) = f(x) + f(y)$ for all numbers x and y , and $f(0) = 0$. This implies that $f(-x) = -f(x)$ and that $f(nx) = n \cdot f(x)$ for any number x and any integer n . For a unit fraction $1/n$, $f(1/n) = (1/n)f(1)$, and thus for a rational number q , it obtains that $f(q) = qf(1)$. Therefore $f(x) = f(1) \cdot x$ for any number x because f is continuous and any number is the limit of a sequence of rational numbers. Letting $\lambda = f(1) = \log(E(1))$ confirms the statement. \square