9 Significance of the Derivative

Theorem 9.1. *Let f be a real valued function on an open set $U \subset \mathbf{R}$. If f attains a maximum or a minimum at a point a in U and f is differentiable at a, then $f'(a) = 0$.*

Proof. If f is differentiable at a, there is a function ε defined on U, such that $\lim_{x \to a} \varepsilon(x) = 0$, and that $f(x) - f(a) = [f'(a) + \varepsilon(x)](x - a)$ for all x in U.

Suppose that $f'(a) \neq 0$. Then $|\varepsilon(x)| < |f'(a)|/2$ for all x in some interval $(a - \delta, a + \delta) \subset U$ or $-|f'(a)|/2 < \varepsilon(x) < |f'(a)|/2$. Therefore,

$$f'(a) - |f'(a)|/2 < f'(a) + \varepsilon(x) < f'(a) + |f'(a)|/2$$

Since $|f'(a)|$ is either $f'(a)$ or $-f'(a)$, the two extremes of the inequality are $f'(a)/2$ and $3f'(a)/2$, both of the same sign as $f'(a)$. This gives rise to a contradiction if f attains a maximum or a minimum at a. For in this case, $f(x) - f(a)$ does not change sign for all x in an interval around a, but $x - a$ does. □

Theorem 9.2 (Rolle's Theorem)**.** *If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there is a number x in (a, b) such that $f'(x) = 0$.*

Proof. Because f is continuous on $[a, b]$, it attains a maximum value and a minimum value on $[a, b]$. If one of these extreme values is attained at a point x in (a, b), then $f'(x) = 0$. Otherwise the maximum and minimum values of f are attained at the endpoints a and b, and because $f(a) = f(b)$, the maximum and minimum values are equal, and so f is constant, so $f'(x) = 0$ for all x in (a, b). □

Theorem 9.3 (Mean Value Theorem)**.** *If f is continuous on $[a, b]$ and differentiable on (a, b), then $f(b) - f(a) = f'(x)(b - a)$ for some number x in (a, b).*

Proof. The function $h(x) = f(x) - \dfrac{f(b) - f(a)}{b - a}(x - a)$ is continuous on $[a, b]$, differentiable on (a, b), and $h(a) = f(a) = h(b)$. By Rolle's Theorem 9.2, $h'(x) = 0$ for some x in (a, b). Now compute $h'(x)$. □

Corollary 1. *If f and g are defined on the same open interval I, and $f'(x) = g'(x)$ for all x in I, then there is a number c such that $g(x) = f(x) + c$ for all x in I. In particular, if $f'(x) = 0$ for all x in I, then f is constant on I.*

Proof. Suppose that $f'(x) = 0$ for all x in I. If $a < b$ are in I, then, by Theorem 9.3 applied to f on $[a, b]$, $f(b) - f(a) = f'(x)(b - a)$ for some x in (a, b); the hypothesis $f'(x) = 0$ forces $f(a) = f(b)$. □

Definition 9.1. If $U \subset \mathbf{R}$ is an open subset and $f : U \to \mathbf{R}$ is a function such that $f'(x)$ exists for all x in U, then f is said to be differentiable on U. Then $f' : U \to \mathbf{R}$ is a function on U, called the derivative of f.

Theorem 9.4 (Taylor's Theorem)**.** *Let U be an open interval in $\mathbf{R}$ and let the function $f : U \to \mathbf{R}$ be n-times differentiable. For any a and b in U, there is a number c between a and b such that*

$$f(b) = f(a) + \frac{1}{1!}f'(a)(b - a) + \frac{1}{2!}f''(a)(b - a)^2 + \cdots + \frac{1}{n!}f^{(n)}(c)(b - a)^n.$$

Proof. Let $R_{n,b}(x) = f(b) - \left[f(x) + \dfrac{f'(x)(b - x)}{1!} + \cdots + \dfrac{f^{(n-1)}(x)(b - x)^{n-1}}{(n - 1)!}\right]$. Then the function $x \mapsto R_{n,b}(x)$ is differentiable on U and $R'_{n,b}(x) = -\dfrac{f^{(n)}(x)(b - x)^{n-1}}{(n - 1)!}$. If $b \neq a$, then Theorem 9.2 applies to $g(x) = R_{n,b}(x) - \left(\dfrac{b - x}{b - a}\right)^n R_{n,b}(a)$ on the interval with endpoinst a and b: there is c between a and b such that $g'(c) = 0$, or $R_{n,b}(a) = \dfrac{1}{n!}f^{(n)}(c)(b - a)^n$. □

Definition 9.2. A real valued function on a subset U of $\mathbf{R}$ if called increasing if, whenever a and b are in U and $a < b$, then $f(a) \leq f(b)$. It is called strictly increasing if $f(a) < f(b)$ whenever $a < b$.

Corollary 2. *If $f'(x) \geq 0$ (respectively, $f'(x) > 0$) for all x on an open interval, then f is increasing (respectively, strictly increasing) on that interval.*

Proof. Let $a < b$ be points in the interval. Theorem 9.3 implies that $\dfrac{f(b) - f(a)}{b - a} = f'(x) \geq 0$, so $f(b) \geq f(a)$. □