8 The Derivative

Definition 8.1. Let f be a real valued function defined on an open subset U of $\mathbf{R}$. Let $a \in U$. The function f is said to be differentiable at a if

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists. It it exists, this limit is denoted by $f'(a)$, and is called the derivative of f at a.

Theorem 8.1. *Let f be a real valued function defined on an open subset $U \subset \mathbf{R}$, and let $a \in U$. Then f is differentiable at a with derivative $f'(a) = A$ if and only if there exist a function ε defined on U with $\lim_{x\to a} \varepsilon(x) = 0$ and such that $f(x) - f(a) = [A + \varepsilon(x)](x - a)$ for all x in U.*

Proof. If f is differentiable at a, then let $\varepsilon(x) = \dfrac{f(x) - f(a)}{x - a} - f'(a)$. Then ε is defined for all $x \neq a$ in U and $\lim_{x\to a} \varepsilon(x) = 0$. Conversely, if such function ε exists, the the limit $\lim_{x\to a} \dfrac{f(x) - f(a)}{x - a} = \lim_{x\to a}(A + \varepsilon(x)) = A$ exists. □

Theorem 8.2. *Let f be real valued on an open subset $U \subset \mathbf{R}$, and let $a \in U$. If f is differentiable at a, then f is continuous at a.*

Proof. If $f'(a)$ exists, then apply Theorem 8.1 to obtain a function ε such that $f(x) - f(a) = [f'(a) + \varepsilon(x)](x - a)$. Then $\lim_{x\to a} \varepsilon(x) = 0$, and thus $\lim_{x\to a} f(x) - f(a) = 0$. □

Theorem 8.3. *Suppose that f and g are differentiable at a. Then $f \pm g$, $f \cdot g$ and, if $g(a) \neq 0$, f/g are all differentiable at a. Their derivatives at a are given by:*

(a) $(f \pm g)'(a) = f'(a) \pm g'(a)$.

(b) $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.

(c) $\left(\dfrac{f}{g}\right)'(a) = \dfrac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$.

Proof. (b) This is based on the identity $\dfrac{f(x)g(x) - f(a)g(a)}{x - a} = \dfrac{(f(x) - f(a))g(a)}{x - a} + \dfrac{f(x)(g(x) - g(a))}{x - a}$ and on the continuity of f and g at a.

(c) Because g is continuous at a, if $g(a) \neq 0$, then $1/g$ is defined on an interval centered at a, and $\dfrac{1/g(x) - 1/g(a)}{x - a} = \left(\dfrac{g(x) - g(a)}{x - a}\right)\left(\dfrac{-1}{g(x)g(a)}\right)$ has limit $-g'(a)/g(a)^2$ as $x \to a$. For f/g, apply (b) to the product $f \cdot (1/g)$.

□

Theorem 8.4. *Let $f : U \to V$ and $g : V \to \mathbf{R}$. If f is differentiable at a, and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a, and $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.*

Proof. Because f is differentiable at a and g is differentiable at $b = f(a)$,

$$f(x) - f(a) = [f'(a) - \varepsilon(x)](x - a) \quad \text{and} \quad g(y) - g(b) = [g'(b) - \eta(y)](y - b)$$

where $\varepsilon(x) \to 0$ as $x \to a$ and $\eta(y) \to 0$ as $y \to b$. Then,

$$\begin{aligned}
(g \circ f)(x) - (g \circ f)(a) &= g(f(x)) - g(f(a)) \\
&= [g'(b) + \eta(f(x))](f(x) - b) \\
&= [g'(b) + \eta(f(x))][f'(a) + \varepsilon(x)](x - a) \\
&= [g'(b)f'(a) + (g'(b)\varepsilon(x) + \eta(f(x))f'(a) + \eta(f(x))\varepsilon(x))](x - a)
\end{aligned}$$

Since f is continuous at a and $\lim_{y\to b} \eta(y) = 0$, the composition $\eta \circ f$ satisfies $\lim_{x\to a} \eta(f(x)) = 0$. Therefore, $(g'(b)\varepsilon(x) + \eta(f(x))f'(a) + \eta(f(x))\varepsilon(x)) \to 0$ as $x \to a$, showing that $g \circ f$ is differentiable at a and $((g \circ f)'(a) = g'(f(a))f'(a)$, by virtue of Theorem 8.1. □