5 Continuous Functions

Definition 5.1. Let $f : \mathrm{dom}(f) \subset \mathbf{R}^k \to \mathbf{R}^l$ be a function and let p be a point in $\mathrm{dom}(f)$. The function f is continuous at p if either the limit $\lim_{x \to p} f(x) = f(p)$, or if p is not a cluster point of $\mathrm{dom}(f)$.

Theorem 5.1. *If f and g are continuous at p, then*

(a) *$f + g$ is continuous at p,*

(b) *$f \cdot g$ is continuous at p.*

Moreover, if g is real valued and $g(p) \neq 0$, then

(c) $1/g$ is continuous at p.

Theorem 5.2. *A function f is continuous at p if and only if whenever p_n converges to p, the sequence $f(p_n)$ converges to $f(p)$.*

Proof. Apply Theorem 4.2 □

Definition 5.2. If f and g are functions, define the composition $g \circ f$ to be the function $g \circ f(x) = g(f(x))$. For this to make sense, the point x must be in $f^{-1}(\mathrm{dom}(g))$, for then x is in the domain of f and $f(x)$ is in the domain of g.

Theorem 5.3. *If g is continuous at p and f is continuous at $g(p)$, then the composition $f \circ g$ is continuous at p.*

Proof. For any given $\varepsilon > 0$ a number $\delta > 0$ id to be found so that, if $|x - p| < \delta$, then $|f(g(x)) - f(g(p))| < \varepsilon$. First use continuity of f to estimate how close $g(x)$ must be to $g(p)$ for this inequality to hold. Since f is continuous at $g(p)$, there is $\delta' > 0$ such that, if $|y - g(p)| < \delta'$, then $|f(y) - f(g(p))| < \varepsilon$. Take $\varepsilon = \delta'$ for continuity of g at p. There is δ such that, if $|x - p| < \delta$, then $|g(x) - g(p)| < \delta'$.

A shorter proof is available using the sequential characterization of continuity (Theorem 5.2). If p_n converges to p, then $g(p_n)$ converges to $g(p)$ and thus $f(g(p_n))$ converges to $f(g(p))$. □

Definition 5.3. A function f is said to be continuous on a set S if f is continuous at p for all p in S. In particular, f is said to be continuous if it is continuous at every point in its domain.

Theorem 5.4. *A function $f : \mathrm{dom}(f) \subset \mathbf{R}^k \to \mathbf{R}^l$ is continuous if and only if for any open set $V \subset \mathbf{R}^l$, there is an open set $U \subset \mathbf{R}^k$ such that $f^{-1}(V) = U \cap \mathrm{dom}(f)$.*

Proof. Note that $f : \mathrm{dom}(f) \to \mathrm{im}(f)$ is onto, that $f^{-1}V = \{x \in \mathrm{dom}(f) \mid f(x) \in V\}$, and that $\mathrm{dom}(f) = f^{-1}(\mathrm{im}(f))$.

If p is in $f^{-1}(V)$, then $f(p)$ is in V and, because V is open, there is $\varepsilon > 0$ such that the ball $B(f(p), \varepsilon) \subset V$. Since f is continuous at p, for this $\varepsilon > 0$ there exists $\delta > 0$ such that if x is in $B(p, \delta) \cap \mathrm{dom}(f)$ then $f(x)$ is in $B(f(p), \varepsilon)$; that is $B(p, \delta) \subset f^{-1}(V)$.

Let $U = \bigcup_{p \in f^{-1}(V)} B(p, \delta)$ be the union of all the balls $B(p, \delta)$, for all p in $f^{-1}(V)$, as obtained in the previous paragraph. The set U is open because it is a union of open sets. Moreover, if p is in $f^{-1}V$, then p is in $\mathrm{dom}(f)$ and in U, as is the center of one of the balls $B(p, \delta)$ used to contruct U; hence $f^{-1}(V) \subset U \cap \mathrm{dom}(f))$. Moreover, if x is in $U \cap \mathrm{dom}(f)$, then x is in some $B(p, \delta) \cap \mathrm{dom}(f)$ for some p in $f^{-1}(V)$, and thus $f(x) \in V$, showing that also $U \cap \mathrm{dom}(f) \subset f^{-1}(V)$.

To prove the converse, let p be in $\mathrm{dom}(f)$ and $\varepsilon > 0$. Apply the hypothesis of the theorem to the open set $V = B(f(p), \varepsilon)$ to obtain an open set U such that $U \cap \mathrm{im}(f) = f^{-1}(V)$. Since p is in U and U is open, there is $\delta > 0$ such that $B(p, \delta) \subset U$, and thus $f(B(p, \delta) \cap \mathrm{im}(f)) \subset V = B(f(p), \varepsilon)$.

□

Definition 5.4. A function f is said to be uniformly continuous on a set S if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if p and q are in S and $|p - q| < \delta$, then $|f(p) - f(q)| < \delta$.