

7 Connected sets

Definition 7.1. A set $S \subset E$ is *disconnected* if there exist two open sets U and V such that

- (a) $S \cap U$ and $S \cap V$ are both nonempty,
- (b) $S \cap U$ and $S \cap V$ are disjoint, and
- (c) $S = (S \cap U) \cup (S \cap V)$.

A set is *connected* if it is not disconnected.

Note 7.1. A subset $I \subset \mathbf{R}$ of real numbers is an interval if and only if whenever x and y are in I and z is a number such that $x < z < y$, then z is also in I .

Theorem 7.1. An interval of real numbers is connected.

Proof. Let $I \subset \mathbf{R}$ be an interval and suppose I is disconnected, so that there exist two open sets U and V satisfying (a), (b) and (c) in Definition 7.1.

Let $a \in U \cap I$ and $b \in V \cap I$, and assume that $a < b$ (why is $a \neq b$?). The intersection $[a, b] \cap V$ is compact because it is closed (V is open) and bounded. It is also non-empty because it contains a . Indeed, a is in $I \cap U$, hence a is not in $I \cap V$, thus a is not in V , so a is in $\mathbb{C}V$. Let $a_1 = \text{l. u. b.}[a, b] \cap \mathbb{C}V$ be the largest number in $[a, b] \cap \mathbb{C}V$. Then $a_1 < b$ because b is in V , thus not in $\mathbb{C}V$. The set $[a_1, b] \cap U$ is compact and non-empty because it contains b . Let $b_1 = \text{g. l. b.}[a_1, b] \cap \mathbb{C}U$ be the smallest number in $[a_1, b] \cap \mathbb{C}U$.

Then $a_1 < b_1$, for if $a_1 = b_1$, then $I \cap U \cap V \neq \emptyset$, contradicting (b) in Definition 7.1. Let c be any number such that $a_1 < c < b_1$. Then c is in I because I is an interval that contains both a_1 and b_1 . It is also in $U \cap V$: if c not V , then c is in $[a, b] \cap \mathbb{C}V$, contradicting that a_1 is the largest number in $[a, b] \cap \mathbb{C}V$; and similarly, if c is not in U , then c is in $[a_1, b] \cap \mathbb{C}U$, contradicting that b_1 is the smallest number in $[a_1, b] \cap \mathbb{C}U$. This implies that c is in $I \cap U \cap V$, contradicting (b) in Definition 7.1 above. \square

Theorem 7.2. A connected subset of real numbers is either an interval (or a singleton, or empty).

Proof. If $x < y$ are any two numbers in S and S is not an interval, then there is a number z not in S such that $x < z < y$. Then open sets $U = (-\infty, z)$ and $V = (z, \infty)$ serve to verify that S is disconnected: $S \cap U \cap V = \emptyset$, $S \subset U \cup V$, and neither $S \cap U$ nor $S \cap V$ is empty. \square

Theorem 7.3. If f is continuous and $\text{dom}(f)$ is connected, then $\text{im}(f)$ is connected.

Proof. Suppose that $\text{im}(f)$ is disconnected. Let U and V be open sets as in Definition 7.1. Because f is continuous, there are open sets U_1 and V_1 such that $U_1 \cap \text{dom}(f) = f^{-1}(U \cap \text{im}(f))$ and $V_1 \cap \text{dom}(f) = f^{-1}(V \cap \text{im}(f))$. These open sets U_1 and V_1 witness that S is disconnected: $U_1 \cap \text{dom}(f)$ and $V_1 \cap \text{dom}(f)$ are both nonempty and disjoint because $U \cap \text{im}(f)$ and $V \cap \text{im}(f)$ are non-empty and disjoint. Moreover, if x is in $\text{dom}(f)$, then $f(x)$ is in one of $U \cap \text{im}(f)$ or $V \cap \text{im}(f)$, and therefore, x is in one of $f^{-1}(U \cap \text{im}(f))$ or $f^{-1}(V \cap \text{im}(f))$ \square

Theorem 7.4 (Intermediate Value Theorem). Let f be a continuous real valued function defined on an interval $[a, b]$. Then f attains all values between $f(a)$ and $f(b)$.

Proof. The set $f[a, b]$ is connected. If $f(a) = f(b)$ there is nothing to prove. If $f(a) < f(b)$, and y is a number such that $f(a) < y < f(b)$, then y is in $f[a, b]$. Therefore, there is x in $[a, b]$ such that $f(x) = y$. Note that x is in (a, b) : if $x = a$, then $y = f(x) = f(a)$, and similarly if $x = b$. \square