6 Compact Sets

Definition 6.1. A set S is bounded if it is contained in some ball; that is, f there exists $R > 0$ such that $S \subset B(0, R)$.

A sequence $p_1, p_2, p_3, \cdots$ is bounded if the set of points $\{p_1, p_2, \cdots\}$ is bounded.

Theorem 6.1. *Let $S \subset \mathbf{R}$ be a nonempty closed set of real numbers. If S is bounded from above, then it has a greatest element. If it is bounded from below, then it has a least element.*

Proof. Because $S \neq \emptyset$, if S is bounded from above, then it has a least upper bound. If $\alpha = \text{l. u. b.}\, S$ but $\alpha \notin S$, then $\alpha \in \complement S$. This subset is open because S is closed. Thus there is $\varepsilon > 0$ such that the open ball $B(\alpha, \varepsilon) \subset \complement S$. (This open ball is the open interval $(\alpha - \varepsilon, \alpha + \varepsilon)$.) This implies that $\alpha - \varepsilon$ is an upper bound for S. Indeed, if $s \in S$ was $> \alpha - \varepsilon$, then $s \in (\alpha - \varepsilon, \alpha + \varepsilon)$, and thus the interval $(\alpha - \varepsilon, \alpha + \varepsilon)$ would not be contained in $\complement S$, contradicting that $\alpha = \text{l. u. b.}\, S$. □

Theorem 6.2. *Suppose that S is a bounded subset of E. Then any sequence $p_1, p_2, \cdots$ of points of S has a subsequence that converges to a point p in E (but perhaps not in S).*

Definition 6.2. A subset S of E is compact if every sequence in S has a subsequence that converges to a point in S.

Theorem 6.3. *A subset S of E is compact if and only if S closed and bounded.*

Proof. Assume that S is compact. If S is not closed, then there is an cluster point p of S such that $p \notin S$. By one of characterizations of cluster points, there is a sequence $p_1, p_2, \cdots$ of points of S that converges to p.

If S is not bounded, then for any positive integer n there is a point p_n in S such that $|p_n - 0| > n$. This sequence p_n cannot contain a convergent subsequence. Indeed, if p_{n_k} is a convergent subsequence, then this subsequence would be bounded, and therefore it will be contained in a ball $B(0, R)$ for some $R > 0$; that is, $|p_{n_k}| < R$ for all n_k, contradicting that $|p_{n_k}| > n_k$.

Assume that S is closed and bounded. Let $p_1, p_2, \cdots$ be a sequence in S. Then, because S is bounded, p_n has a subsequence that converges to a point p in E. Because S is closed, this point p must belong to S. □

Corollary 6.1. *The closure of a bounded subset of E is compact.*

Theorem 6.4. *If f is continuous and K is a compact set in* $\text{dom}(f)$*, then $f(K)$ is a compact set.*

Proof. Let a_n be a sequence in $f(K)$. Then $a_n = f(p_n)$ for some p_n in K. Because K is compact, the sequence $p_1, p_2, \cdots$ has a subsequence $p_{n_1}, p_{n_2}, \cdots$ that converges to a point p in K. Because f is continuous at p, the subsequence $a_{n_j} = f(p_{n_j})$ converges to $f(p)$, a point in $f(K)$. □

Corollary 6.2. *If f is real valued on a non-empty compact set K, then f attains a maximum and a minimum value on K.*

In particular, if f is continuous on an interval $[a, b]$, then there is a number x_M such that $a \leq x_M \leq b$ and $f(x) \leq f(x_M)$ for all x in $[a, b]$, and similarly, there is a number x_m such that $a \leq x_m \leq b$ and $f(x) \geq f(x_m)$ for all x in $[a, b]$.

Proof. By Theorem 6.4, $f(K)$ is a compact set of real numbers. Since it is non-empty, it has a largest element y_M (maximum) and a smallest element y_m (a minimum). Because y_M and y_m are in the image $f(K)$, there are points x_M and x_m in K such that $f(x_M) = y_M$ and $f(x_m) = y_m$. □

Theorem 6.5. *A continuous function is uniformly continuous on any compact subset of its domain.*

Proof. Let f be continuous and $K \subset \text{dom}(f)$ be compact. If f is not uniformly continuous on K, then there exists $\varepsilon > 0$ such that for any $n = 1, 2, 3, \cdots$ there are points p_n and q_n in K with $|p_n - q_n| < \dfrac{1}{n}$ and $|f(p_n) - f(q_n)| \geq \varepsilon$. The sequence p_n has a subsequence p_{n_i} that converges to a point p in K, and the sequence q_{n_i} has a subsequence that converges to a point q in K. Then the subsequence $p_{n_{i_j}}$ converges to p and the subsequence $q_{n_{i_j}}$ converges to q. Then $p = q$ because $p_n - q_n| < \dfrac{1}{n}$ for all n, but $f(p) \neq f(q)$ because $|f(p_n) - f(q_n)| \geq \varepsilon$ for all n. □