Math 311. Handout 9. 10/1/08 **Name:** ____________________________

Regular polygons Regular polygons with three sides (equilateral triangle), four sides (square), and six sides are easy to construct.

1 (Square)**.** To construct a square inscribed in a circle, draw two perpendicular diameters to the given circle. These diameters will intersect the circle at the vertices of the square.

2 (Hexagon)**.** To construct a regular hexagon inscribed in a circle O^B: (a) draw the circle B^O, (b) let A and C be the points of intersection of O^B and B^O (c) with the point of the compass at C, draw the circle C^O, obtaining a new point D (d) with the point of the compass at D, draw the circle D^O, obtaining E, and (e) with the point at E, draw the circle E^O, obtaining a new point F.

The points A, B, C, D, E, and F are the vertices of the required regular hexagon

3 (Equilateral triangle)**.** To construct an equilateral triangle inscribed in circle, first construct and inscribed regular hexagon with vertices A, B, C, D, E, and F. Then A, C and E (for example) are the vertices of an inscribed equilateral triangle.

The Pentagon There are several methods of constructing a regular pentagon with ruler and compass.

4. Given a circle with center O through A, follow these steps to construct a regular pentagon inscribed in O^A.

(a) Let $\overline{AB}$ be the diameter of O^A through A.

(b) Let E be the midpoint of $\overline{OA}$.

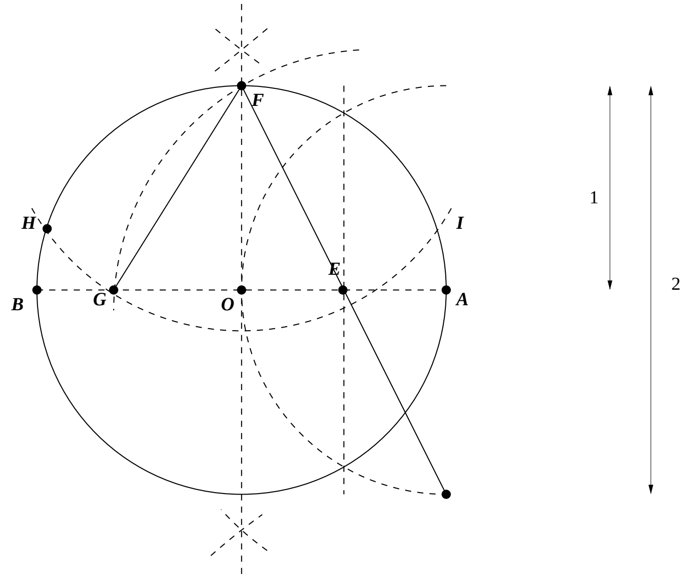
(c) Let F be a point of intersection of O^A and the perpendicular to $\overline{AB}$ at O. (This point F is the first vertex of the pentagon.)

(d) Let G be the intersection of $\overline{AB}$ and E^F.

(e) The circle F^G meets circle O^A at two new vertices H and I of the required pentagon.

(f) Draw two circles with centers H and I and with the same radius FG to find the last two vertices of the required pentagon.

5. The construction performed in Problem 4 produces the following figure. We will compute the length of segment $\overline{FH}$. The radius of the circle $O^A = O^B$ is 1, thus its diameter $AB = 2$.



(a) Use the Pythagorean Theorem on $\triangle OEF$ to prove that $FE = \dfrac{\sqrt{5}}{2}$.

(b) Prove that $OG = \dfrac{\sqrt{5}-1}{2}$.

(c) Use the Pythagorean Theorem on $\triangle GOF$ and the value of OG found in (b) to obtain $FG = \sqrt{\dfrac{5-\sqrt{5}}{2}}$.

6. The next construction of a regular pentagon inscribed in a circle is somewhat similar, but it has the distinctive feature of using only a compass. This is mentioned for historical reasons, as it was a long standing problem to determine if every construction that could be performed with ruler and compass could also be performed with compass alone. This problem was eventually solved by the Italian geometer Mascheroni (1750–1800). The related problem, whether it is possible to perform all "ruler and compass" constructions could also be performed with ruler only, has a negative answer.

In order to construct an inscribed regular pentagon with compass alone, we take as given a circle, O^B, with center O and passing through B. The perform the following steps.

(a) Draw the circle, B^O, with center B passing through O.

(b) Let A and C be the points of intersection of B^O and O^B.

(c) Draw the circle C^O. This circle intersects O^B in B and in a new point D.

(d) Notice that A, B, C, and D are consecutive vertices of a regular hexagon inscribed in the circle O^B.

(e) Draw the circle with A and D as centers and AC as radius. Let X be one of the points of intersection.

(f) The line $\overleftrightarrow{OX}$ and the circle O^B meet at points F and K, with F being the midpoint of $\overline{BC}$.

(g) Draw circle F^O, meeting O^B at G and H.

(h) Let Y be the point at distance OX from both G and H and which is separated from X by O.

(i) The length of the segment AY is equal to a side of the required inscribed regular pentagon.

Other regular polygons It turns out that the next regular polygon in the list, namely, the heptagon, cannot be constructed with ruler and compass alone. This was discovered by Gauss (1777–1855) when he was just 19 year old. In fact, he completely determined precisely which regular n-sided polygons can be constructed with ruler and compass alone. To explain the content of Gauss Theorem, we need to recall prime number (a whole number > 1 which has no other factors but 1 and itself, and a peculiar family of prime numbers called Fermat primes. Fermat primes are prime numbers of the form $F_k = 2^{2^k} + 1$, where $k = 0, 1, 2, \cdots$ The first Fermat prime numbers are

$$\begin{aligned} F_0 &= 2^{2^0} + 1 = 2^1 + 1 = 3 \\ F_1 &= 2^{2^1} + 1 = 2^2 + 1 = 5 \\ F_2 &= 2^{2^2} + 1 = 2^4 + 1 = 17 \\ F_3 &= 2^{2^3} + 1 = 2^8 + 1 = 257 \\ F_4 &= 2^{2^4} + 1 = 2^{16} + 1 = 65537 \end{aligned}$$

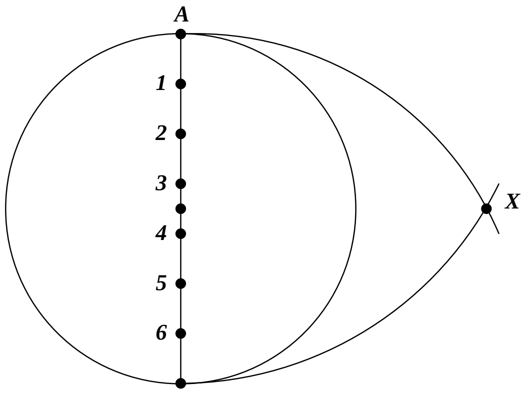
The next number in the list $F_5 = 2^{2^5} + 1 = 4294967297$ is not prime.

Theorem 1 (Gauss). *A regular n-sided polygon can be constructed with ruler and compass alone if and only if all the odd prime factors of n are distinct Fermat primes.*

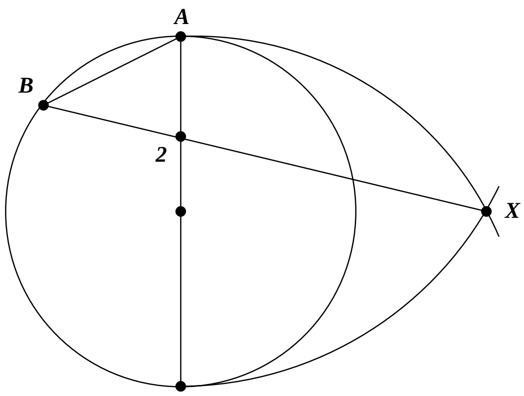
7. List all the regular polygons with up to 100 sides which can be constructed with ruler and compass alone.

8. I was taught in high-school the following construction of a regular polygon with n sides inscribed in a given circle. Here we consider the case of a heptagon, $n = 7$.

(a) Given circle O^A, draw circle from the points of intersection of O^A and $\overleftrightarrow{OA}$ to find point X. Then divide diameter through A into n equal parts (in this case, $n = 7$).



(b) Find the point B where the the ray from X and through the second point in the diameter meets the circle O^A. The point B is one vertex of the required polygon (in this case, heptagon) and the segment $\overline{AB}$ is one of the sides of the required polygon.



(c) Find the next vertex by intersecting the circle with center B and radius AB with the circle O^A, and so on.