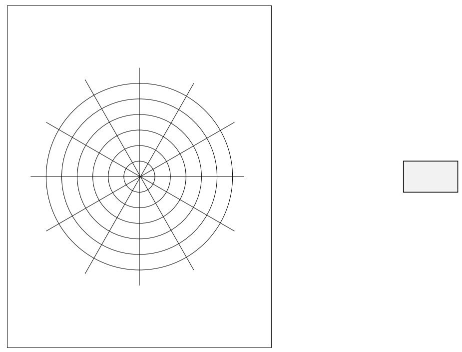
¶ **1** (How to draw a spiral)**.** For this construction you need a sheet of polar graph paper and a business card, like these:



Mark a point P_1 on one of the rays of the polar graph paper. Then place the card so that one side passes through P_1, and slide it so that the adjacent side lies exactly over the next ray. Mark the point on the next which is at the corner of the right angle in your card, and label it with P_2. Now do the same thing but this time starting with the point P_2; now the card touches P_2 with one side, and the adjacent side is exactly over the next ray around. Mark the point where the corner of the card touches this ray, and call it P_3. Continue creating points P_4, P_5, and so on until physically possible, and let P_n denote the last point that you have created. Next join the points P_1, P_2, up to P_n with curve as smooth as you can possible manage.

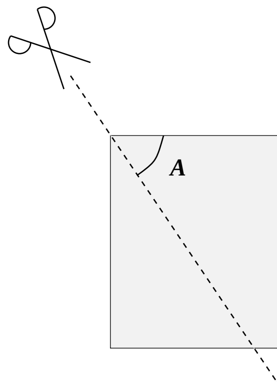
¶ **2.** Most likely your curve spirals in toward the center of the polar graph in a counterclockwise fashion.

(a) Why do you think is the reason for that?

(b) How would you modify the construction so that the curve spirals in in a clockwise fashion?

¶ **3.** Why do the points P_1, P_2, etcetera, spiral in to the center of the polar graph paper? Could they stay at the same distance? (Hint: on a right triangle, the hypotenuse is longer than its legs.)

¶ **4.** Now pick your card and cut off a corner at an angle A of your choice. Like this:



Then repeat the construction that we have done before with this cut-off card. Place a point P_1 on one of the rays, then place the card with one side on P_1 and the neighboring side on the next ray, and find the next point P_2 at the vertex of angle A. Repeat this construction with P_2 in place of P_1 to obtain P_3 on the next ray following that of P_2, and so on.

(a) What kind of curve do you obtain when you join the points $P_1, P_2, \cdots$.

(b) Does it spiral in? out? closes up?

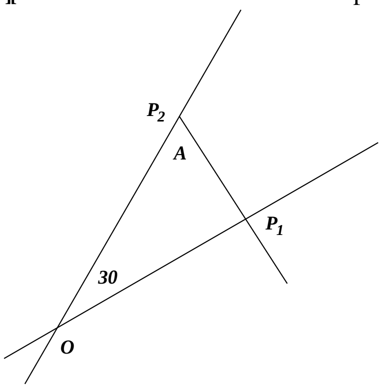
¶ **5.** In fact, anything could happen, depending on the angle A that you cut at the corner of the card. We know that if $A = 90°$, then the curve spirals in (when taking the second ray in the counterclockwise side of the first ray).
Let O be the center of the polar graph paper, and compare the lengths of the segments OP_1 and OP_2.

(a) If $OP_1 < OP_2$, the curve (spirals in) (spirals out) (circles).

(b) If $OP_1 > OP_2$, the curve (spirals in) (spirals out) (circles).

(c) If $OP_1 = OP_2$, the curve (spirals in) (spirals out) (circles).

¶ **6.** It turns out that we can compare the lengths OP_1 and OP_2 in terms of the angle A.



(a) In the figure, what is the angle at P_1?

(b) For what value of A is $OP_1 = OP_2$?

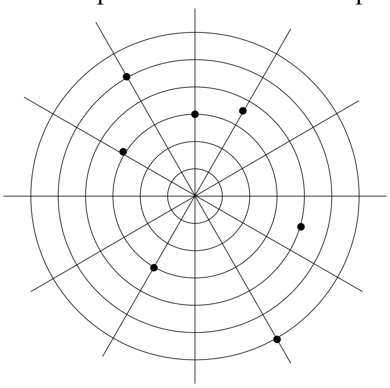
(c) When is $OP_1 > OP_2$?

(d) When is $OP_1 < OP_2$?

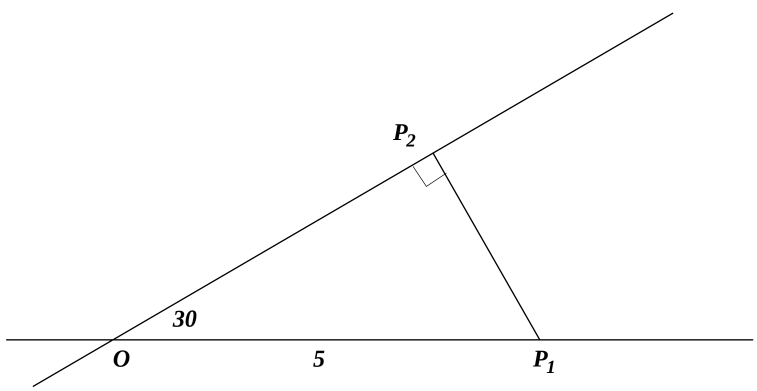
Summarizing,

	bigger than	degrees,		spirals in
If angle A is	equal to	degrees,	then the curve	circles
	bigger than	degrees,		spirals in

¶ **7.** To locate points in the plane we use rectangular coordinates (x, y), based on a pair of perpendicular axis. When using a polar graph paper, it is more appropriate to locate points using **polar coordinates**. These coordinates have the form (r, A), where r is the distance to the origin, and A is the angle that the point makes with the horizontal axis.
Find the polar coordinates of the points marked in this polar graph. (Consecutive circles are at 5 units distance apart.)

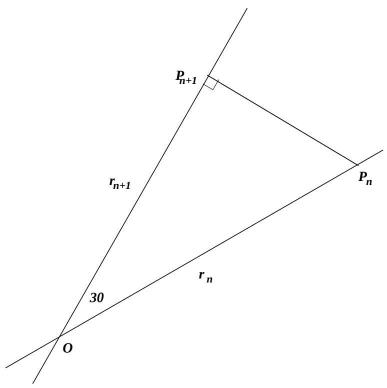


¶ **8.** Going back to the construction of the spiral, the following question arises: if you know the polar coordinates of your initial point P_1, how can you determine the polar coordinated of the next point P_2?
Use the following figure to find out the polar coordinates of P_2 given that P_1 has polar coordinates $(5, 0)$.



¶ 9. Because our construction of the points $P_1, P_2, P_3, \cdots$ was based on rays that make an angle of 30°, we know that if P_n has polar coordinates of the form (r_n, A_n), then the next point P_{n+1} (counterclockwise) will have polar coordinates $(r_{n+1}, A_n + 30)$.

(a) Use the figure below to find the distance r_{n+1} in terms of the distance r_n.

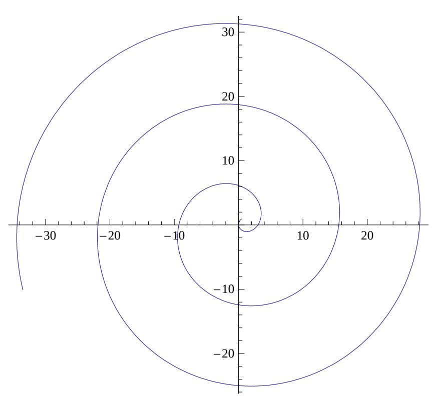


(b) Note the curious property enjoyed by the points $P_{n+1} = (r_{n+1}, A_{n+1})$. The angle $A_{n+1} = 30n$ degrees, and the distance $r_{n+1} = r(\sqrt{3}/2)^n$, where r was the distance from the initial point to the origin. The ratio

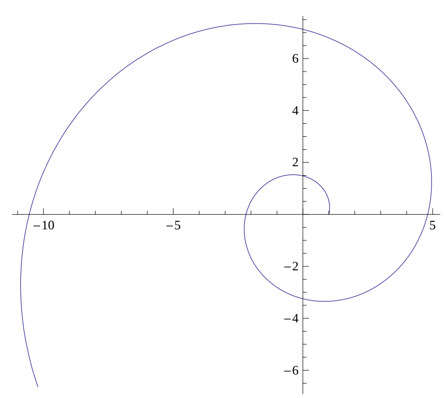
$$\frac{r_{n+1}}{r_n} = \frac{\sqrt{3}}{2}$$

is independent of n.

¶ 10. The Archimedean spiral has polar coordinates (r, θ) of the form $r = a + bA$ where a and b are some fixed numbers. The number a controls the speed, the number b controls the distance between consecutive turnings.
The logarithmic spiral has polar coordinated of the form $r = ab^\theta$, where$a$ and b are some fixed numbers. It has the geometric property that its tangent line makes a constant angle with the radial line at any point.



Archimedean Spiral



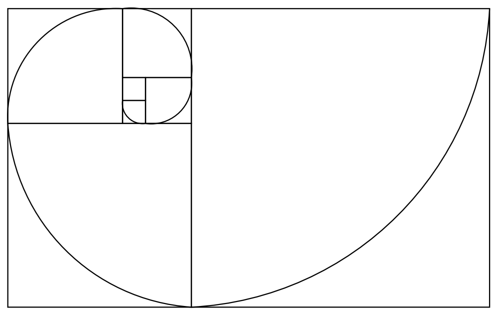
Logarithmic Spiral

Logarithmic spirals can be readily distinguished from Archimedean spirals by the fact that the distances to the origin between consecutive full turns increase in geometric progression, while in the Archimedean spiral these distances increase in arithmetic progression.

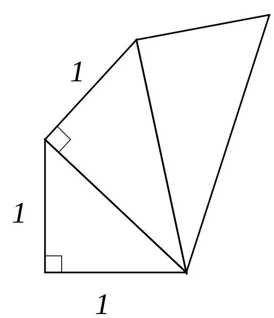
¶ **11.** (a) The sequence $1, 2, 4, 8, \ldots$ gives rise to the polar coordinates $(2^n, 30°n)$, and it has the property that ratio of any two consecutive terms is 2.

(b) The sequence $1, 4, 9, 16, \ldots$ give rise to the polar coordinates $(n^2, 30°n)$. The ratio of any two consecutive terms is $(n+1)^2/n^2$. This is not a constant, but it approaches 1 as n increases without bound.

(c) The sequence $1, 1, 2, 3, 5, 8, \cdots$, where each terms f_n is the sum of the two previous terms $f_n = f_{n-1} + f_{n-2}$ is called the Fibonacci sequence. The ratios f_{n+1}/f_n of consecutive terms of the Fibonacci sequence as n increases without bound approach the Golden ration $\frac{1+\sqrt{5}}{2}$. Even if the ratio f_{n+1}/f_n is not a constant, Fibonacci numbers are related to a logarithmic spiral:



(d) The Pythagorean Spiral, or Spiral of Theodorus, is made up of contiguous right triangles all having one of their legs equal to 1, starting with a right triangle with units length as shown. The distance between consecutive turns is not constant, but it apporaches the number π as the number of turns increases. It thus approximates an Archimedean spiral.



Bibliography

[1] P. Hilton and J. Pedersen, *Mathematical Reflections*, Springer, New York, 1998.