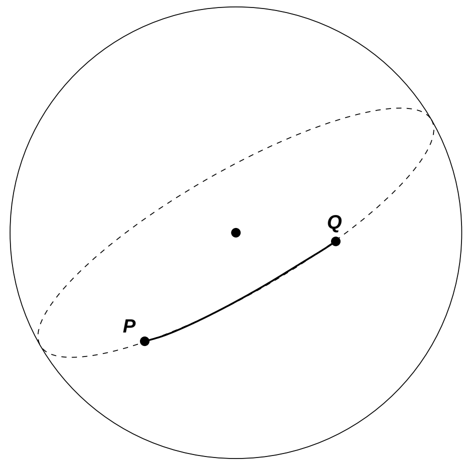
¶ **1.** To understand the geometry of the surface of a sphere we first need to understand how to measure distances on it. In the plane, the distance between two points P and Q is given by the length of the line segment joining P and Q. Thus the curve of shortest length joining P and Q is the line segment PQ.



¶ **2.** On the surface of a sphere, the curve of shortest length joining two points P and Q is the arc of a great circle passing through P and Q.
A great circle is the intersection of the sphere with a plane passing through the center of the sphere. If P and Q are not antipodal points (that is, they are not diametrically opposite points) then there is a unique great circle passing through both P and Q.

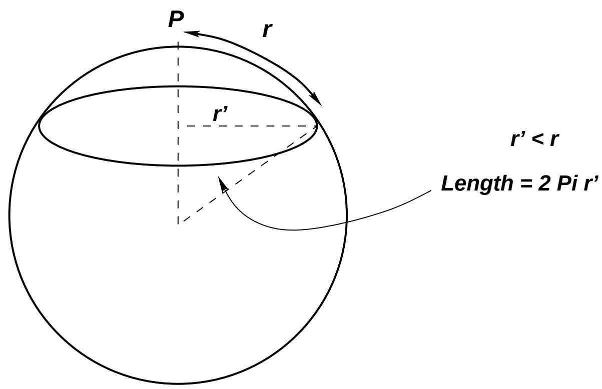
¶ **3.** If the sphere has radius R, then any great circle is a circle of radius R, and therefore it has length πR. Consequently, any two points on the surface of the sphere are at distance $\leq 2\pi R$. They are exactly at distance $= \pi R$ if and only if they are antipodal points.

¶ **4.** Great circles on the surface of the sphere play the same role as straight lines on the plane. Many properties of lines on the plane continue to hold for lines on the sphere. For example, there is a great circle passing through any two points.
But other properties fail: for example,

(a) there are infinitely many lines passing through a pair of antipodal points,

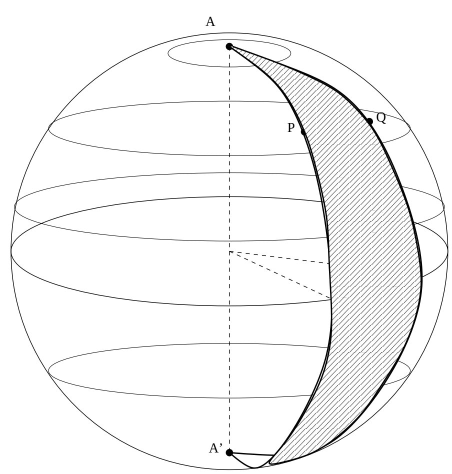
(b) there are no parallel lines because any two great circles intersect.

¶ 5. On the plane, a circle with center P and radius r is the set of points at distance r from P. The length of a circle of radius r is $2\pi r$.
On the surface of a sphere, a circle with center P and radius r can be similarly defined as the set of points at distance exactly r from P. From the following picture we see that the length of such circle is smaller than $2\pi r$.



¶ 6. A tight belt is placed along the equator of the Earth. How much slack do you have to add to this belt so that a cruise ship 50 meters tall can sail across under the belt along the Pacific Ocean from San Diego to Santiago (Chile)?

¶ 7. We can define the angle $\angle PAQ$ to be the portion of the sphere bounded by the arcs of the great circles through P, A and through Q, A. We can define the magnitude of the angle $\angle PAQ$ to be the magnitude of the angle formed by the half planes APA' and AQA' along the segment AA' (here A' is antipodal to A).



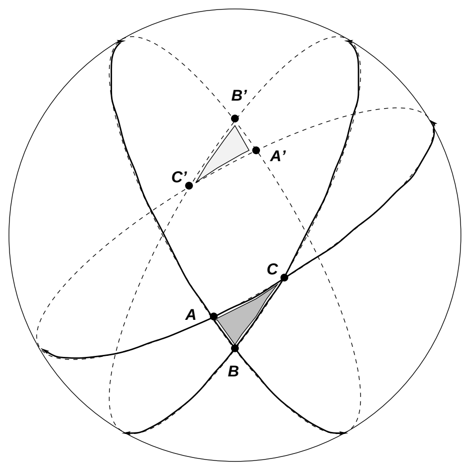
¶ 8. Three points on the surface of the sphere are colinear if they lie on the same great circle. Three non-colinear points together with the arcs of the great circles joining them, determine a spherical triangle. Many properties of triangles in the plane are shared by spherical triangles. For example, in a spherical triangle, the length of any side is smaller than the sum of the lengths of the other two sides. However, other properties do not hold: any three mutually perpendicular planes determine a spherical triangle with three right angles.

¶ 9. The area of the surface of a sphere of radius R is $4\pi R^2$. The area of a sector of the surface of the sphere between two great circles making an angle of A degrees must be proportional to the angle. Therefore, because a the full surface of the sphere is a sector of 360 degrees, we must have

$$\text{Area} = \frac{\angle A}{360} 4\pi R^2.$$

for the area of a sector of A degrees on the surface of a sphere of radius R.

¶ 10. *The sum of the angles of a spherical triangle is greater than* 180 *degrees.*
Let the points A, B, C be the the vertices of the triangle. Then extend the sides of the triangle to three great circles on the surface of the sphere. If A', B', C' are the antipodal points of A, B, C, then the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent.



The two sides of the triangle through the vertex A determine two great circle that intersect at A and at the antipodal A'. These two great circles enclose a region $I(A)$ on the surface of the sphere consisting of two spherical sector of angle $\angle A$, hence of area $2\frac{\angle A}{360}4\pi R^2$.
Similarly, the vertex B and the vertex C determine regions $I(B)$ and $I(C)$ with areas $2\frac{\angle B}{360}4\pi R^2$ and $2\frac{\angle C}{360}4\pi R^2$, respectively.
The three regions $I(A)$ $I(B)$ and $I(C)$ cover the whole surface of the sphere, and they overlap on the triangle $\triangle ABC$ and the antipodal $\triangle A'B'C'$. Therefore

$$\begin{aligned}
\text{Area}\, I(A) + \text{Area}\, I(B) + \text{Area}\, I(C) &= 4\pi R^2 + 4\,\text{Area}\,\triangle ABC \\
2\frac{\angle B}{360}4\pi R^2 + 2\frac{\angle B}{360}4\pi R^2 + 2\frac{\angle B}{360}4\pi R^2 &= 4\pi R^2 + \text{Area}\,\triangle ABC \\
\frac{\angle A + \angle B + \angle C}{180}4\pi R^2 &= 4\pi R^2 + 4\,\text{Area}\,\triangle ABC \\
\frac{\angle A + \angle B + \angle C}{180} &= 1 + \frac{\text{Area}\,\triangle ABC}{\pi R^2} \qquad \text{(Divide both sides by } 4\pi R^2\text{)}
\end{aligned}$$

and therefore

$$\angle A + \angle B + \angle C > 180 \text{ degrees}$$

because $\dfrac{\text{Area}\,\triangle ABC}{\pi R^2} > 0$