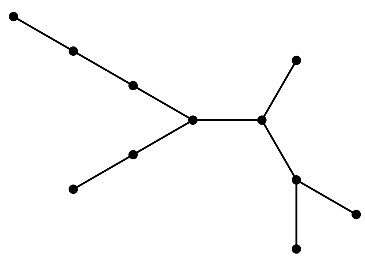
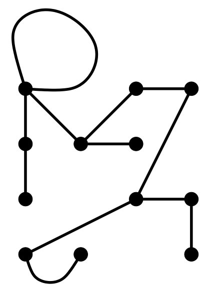
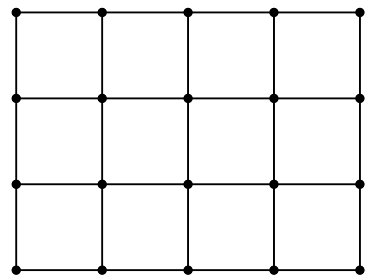
¶ **1.** A simple path in a graph is a path with no repeated edges. A simple circuit is a circuit without repeated edges.

¶ **2.** Trees are special kinds of graphs. A tree is a connected graph with no simple circuits.
Which of the following graphs is not a tree? Why?

¶ **3.** There are many other characterizations of trees. The following properties are equivalent for a graph G.

(a) G is a tree;

(b) G is a connected graph which becomes disconnected upon removal of any edge;

(c) if A and B are distinct vertices in G, then there is exactly one simple path from A to B.

¶ **4.** Draw any tree. Count the number of its vertices (V) and the number of its edges (E).

(a) What is the value of $V - E$? Why? (i.e., can you prove it?

(b) Can you find a (connected) graph which is not a tree but for which the edge-vertex count $V - E = 1$?

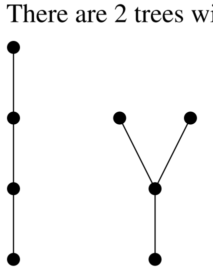
¶ **5.** It is an interesting exercise in visualization to tabulate all the trees with a given number of vertices. For example:

(a) There is 1 tree with exactly 1 vertex:

(b) There is 1 tree with 2 vertices:

(c) There is 1 tree with 3 vertices:

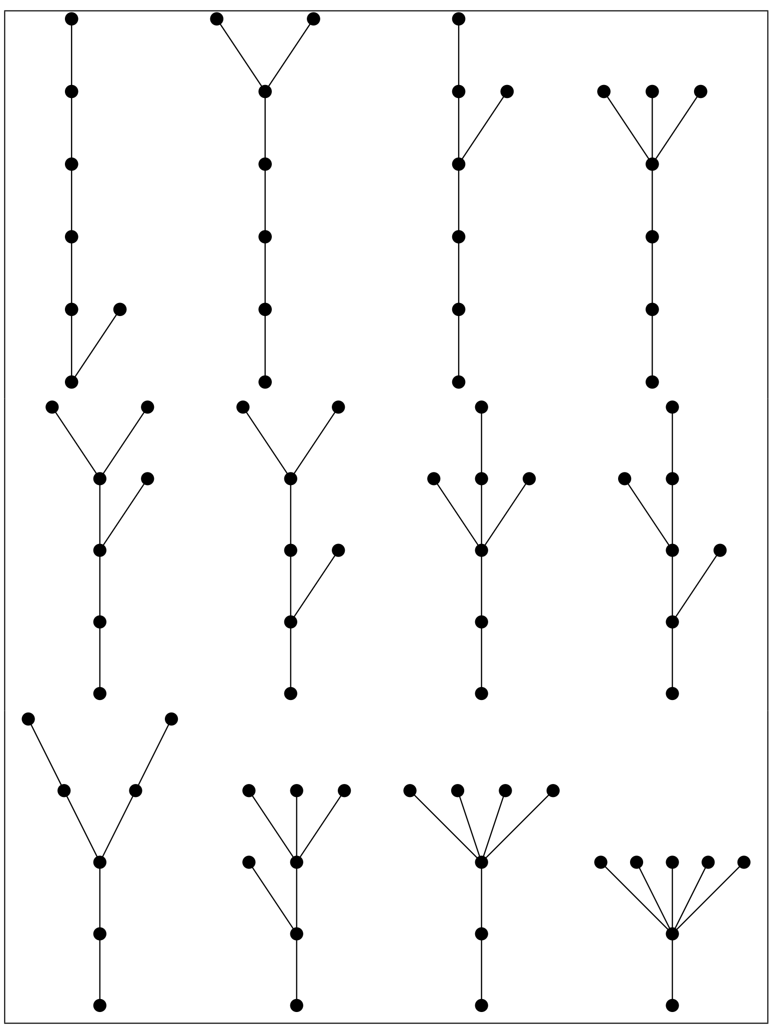
(d) There are 2 trees with 4 vertices:



(e) There are with 5 vertices:

(f) There are with 6 vertices:

¶ 6. There are 11 trees with seven vertices. Twelve are pictured below. Can you find the duplicate?

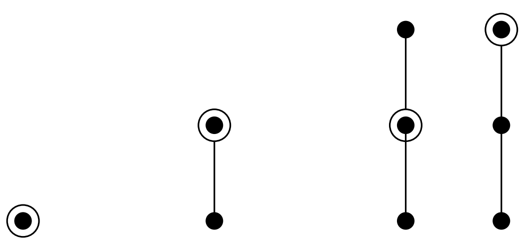


¶ **7.** There is no closed formula for the number of trees with n vertices. Some values are listed in the following table.

Number of vertices	7	8	9	10	11	12
Number of trees	11	23	47	106	235	551

¶ **8.** When trying to make sure if two trees (or graphs) are really different, we use geometric or topological invariants. We have studied several of those: "connected" and "total degree"
In thecase of trees these invariant are not useful. Every tree is connected, and the total degree of a tree with n vertices is $2n - 2$.
Other invariants involve distances. Recall that the distance between two vertices A and B, denoted by $d(A, B)$, in a connected graph is the length of a shortest path from A to B. Note that $d(A, A) = 0$ and that $d(A, B) = d(B, A)$.
The **Wiener index** of a graph G, denoted y $W(G)$, is the sum of the between pairs of vertices of G.
Compute the Wiener index of all trees with 5 vertices.

¶ **9.** Many problems involve special kind of trees. A rooted tree is a tree with a distinguished vertex, called the root. Two rooted trees are equivalent when there is an equivalence between their underlying trees that preserves the root. Below you have 1 rooted tree with one vertex, 1 rooted tree with 2 vertices, and 2 rooted trees with 3 vertices.

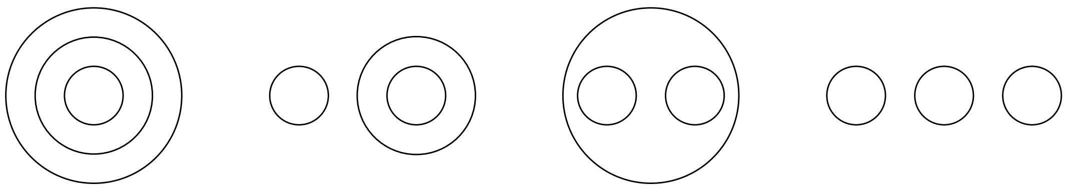


There are four different rooted trees with four vertices. Can you find the other 2?

¶ **10.** The number of rooted trees with n vertices is given in the following table:

Number of vertices	2	3	4	5	6	7	8
Number of rooted trees	1	2	4	9	20	48	115

¶ **11.** Find the number of ways in which n circles in the plane can be internal or external to one another. The picture below show the 4 different ways in which 3 circle can be so arranged. To each arrangement there corresponds one of the 4 different rooted trees with 4 vertices.



¶ **12.** The following problem appears to have been posed by the French lawyer and mathematician Pierre de Fermat (1601-1665) to the Italian physicist Evangelista Torricelli (1608–1647). Torricelli's student Vicenzo Viviani (1622–1703) contributed to solving this problem.

> *In the plane of a triangle find a point for which the sum of distances to the vertices of the triangle is smallest.*

This problem and others of its kind are known today as Steiner problem, after the German geometer Jakob Steiner (1796–1863).

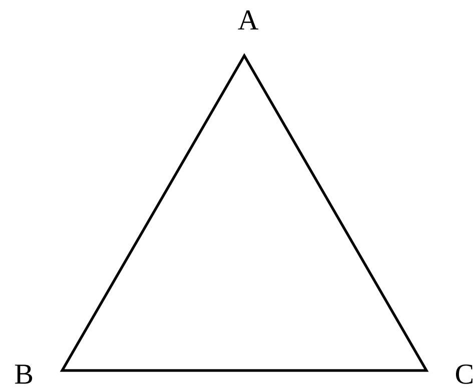
Pierre de Fermat

Evangelista Torricelli

Vicenzo Viviani

Jakob Steiner

¶ **13.** Suppose that A, B, C are the vertices of an equilateral triangle with side length s, as in the figure.



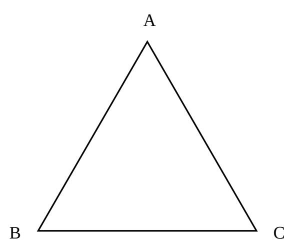
(a) What is the height of the triangle $\triangle ABC$?

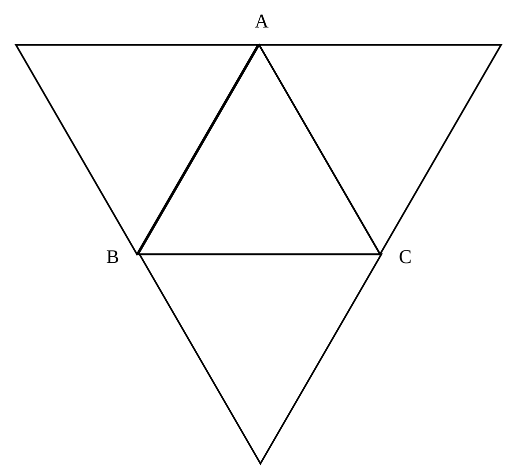
(b) Pick any point P inside the triangle and let x, y, and z be the distances from P to each of the sides AB, BC, and CA. What is $x + y + z$?

(c) What point P would you chose so the the sum of the distances from P to each of the sides of $\triangle ABC$ is smallest? (The distance from a point to a line is the length of the perpendicular segment from that point to the line.)

¶ **14.** What we have discovered in the previous problem is known as Viviani's Theorem: *The sum of the distances from a point inside an equilateral triangle to the sides of the triangle is equal to the altitude of the triangle.*
Since the distance from a point P to a line ℓ is the smallest distance from P to any point Q on the line ℓ, we can reverse Viviani's theorem and state it as follows: From a point P inside an equilateral triangle $\triangle ABC$ construct three segment to each of the three sides AB, BC and CA. Then the sum of the lengths of this segments is smallest when each segment is perpendicular to the corresponding side of the triangle.
We now analyze Steiner's problem proper for an equilateral triangle $\triangle ABC$ of side s.

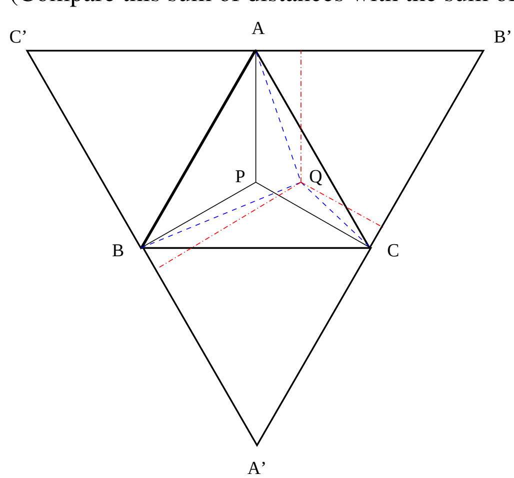
(a) Construct an equilateral triangle on each of the sides of $\triangle ABC$. Label the points A', B', C' as in the figure. What kind of triangle is $A'B'C'$?





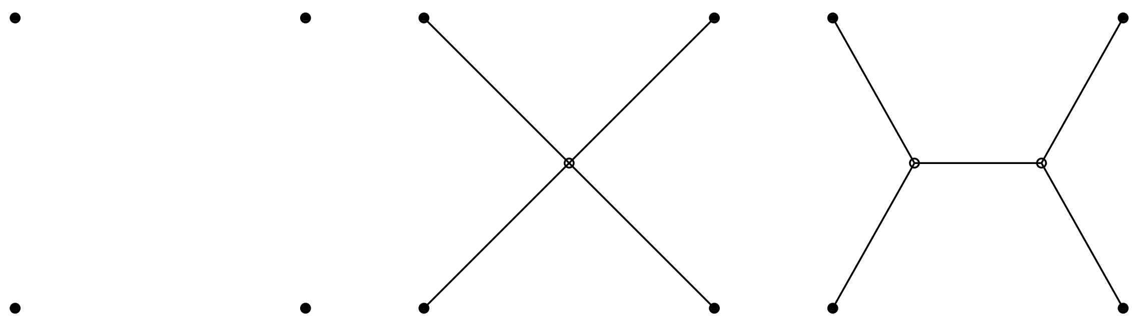
(b) Let P be the point inside the triangle ABC where the perpendicular bisectors to the sides of ABC all meet. What is the relationship between the sum of the distances $AP + BP + CP$ and the triangle $\triangle A'B'C'$?

(c) Pick any other point Q inside $\triangle ABC$. Explain why $AQ + BQ + CQ$ is at least as large than $AP + BP + CP$. (Compare this sum of distances with the sum of the lengths of the red segments.)



¶ **15.** We have seen that the Steiner point of an equilateral triangle is the point where the perpendicular bisector to each of the sides of the triangle meet. Putting it this way, the solution to the equilateral triangle does not completely clarify what may be the general case.
The solution to finding the Steiner point of the triangle $\triangle ABC$ has in two cases: (a) if all the angles of $\triangle ABC$ are smaller than 120°, then the Steiner point is the point inside $\triangle ABC$ subtending an angle of 120° with each side; (b) if $\triangle ABC$ has one vertex with angle greater than or equal to 120°, then that vertex is the Steiner point.

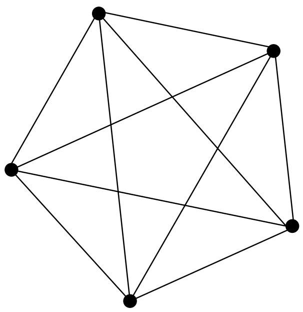
¶ **16.** The Steiner tree problem is to find a minimal length network that spans a set of point in the plane while allowing for the addition of auxiliary points. The case we have examined, that of three points, is the simplest non-trivial instance of the problem.
Consider a set of four points making the vertices of a square of side 1, like in the figure on the left below. What is the Steiner network for this set?



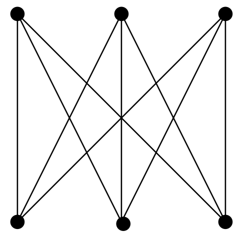
(a) Suppose that you construct a network joining the corners of the square to its center, as in the figure in the middle. What is the length of this network?

(b) Suppose that you construct a network joining the corners of the square to two other points inside the square so that the angles at those points is 120 degrees. What is the length of this network?

¶ **17.** As mentioned in a previous lecture, some graphs can be drawn in the plane in a way that edges cross at vertices, if at all. Those graphs are **planar graphs**. For example, any tree is a planar graph.
There are graphs that are not planar. It is a mathematical theorem that these two graphs are not planar graphs, and in facts, every non planar graph "contains" one of them.

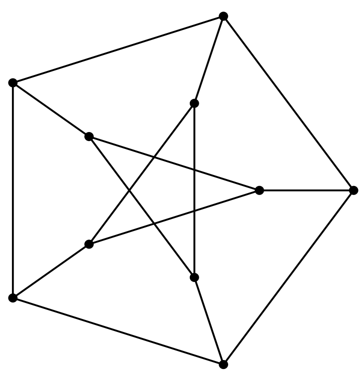


K_5



$K_{3,3}$

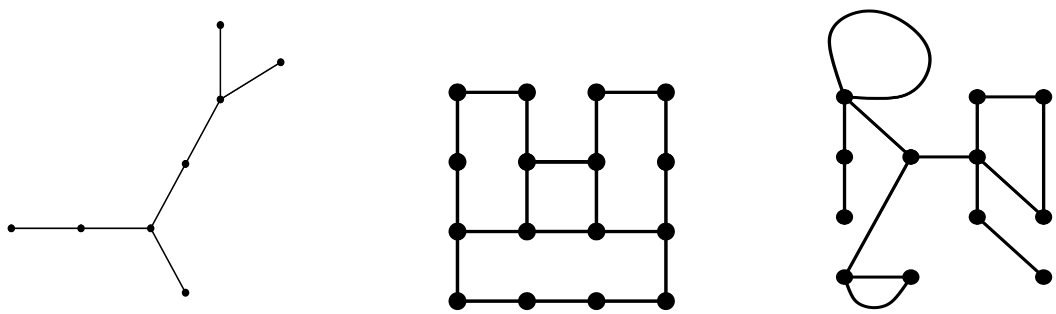
¶ **18.** These graph is called the Petersen Graph. It is a non planar graph. If you study it carefully, you will notice that it "contains" a K_5 graph.



¶ **19.** It turns out that the Euler formula that we have studied earlier can be used to determine if a graph is non-planar. Indeed, if your draw a graph in the plane, then it determines a disk-like decomposition of the surface of the sphere: the edges and vertices of the graph bounds a collection of regions which are the faces of the decomposition.
Therefore, Euler's identity applies:

$$V - E + F = 2.$$

¶ **20.** When we walk around the boundary of a face determined by a planar graph, we encounter a sequence of edges and vertices, finally returning to the starting position. This is in fact a circuit in the graph, and its length is called the degree of the face.
What are the degrees of the faces of these planar graphs?



¶ **21.** For a simple planar connected graph with at lest three vertices, $E \leq 3V - 6$. As we have said, the graph pictured in the plane determines a disk-like decomposition of the surface of the sphere. This decomposition has the same number of vertices V and of edges E as the graph, and it has certain number of faces F. Note that one of the faces, as viewed in the plane, is the outside infinite face.

(a) Group the faces into F_1 of degree 1, F_2 of degree 2, and so on. Then $F = F_1 + F_2 + F_3 + F_4 + \cdots$. Explain why $F_1 = F_2 = 0$ for a connected planar graph with $V \geq 3$.

(b) Explain why $2E = 3F_3 + 4F_4 + \ldots \geq 3F$

(c) Explain why $V - E + F < V - (1/3)E$

(d) Explain why $E \leq 3V - 6$