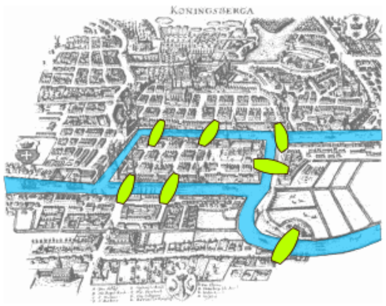
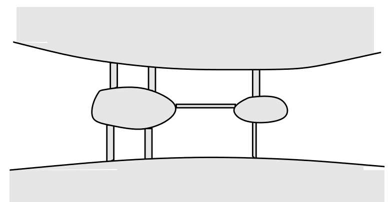
¶ **1.** Graph theory is a broad area of mathematics. Besides their theoretical significance, graphs have many practical applications: computer science, Internet, management science, scheduling.
Euler (1707–1783) is credited with having originated the areas of graph theory. He spent part of his career employed by in the city of Königsberg (now Kaliningrad). The city of Königsberg was divided by the river Pregel into four landmasses connected by bridges as shown in the left figure below. The citizens of Königsberg, as curious people and avid walkers, wondered if it was possible to cross all seven bridges without crossing any bridge more than once.

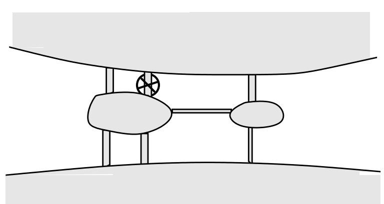


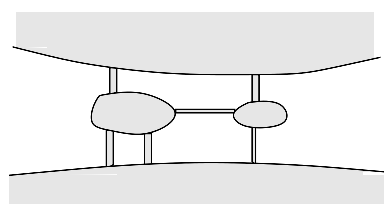


(a) Can you solve the Königsberg Bridge Problem? That is, can you find a path that traverses each bridge exactly once?

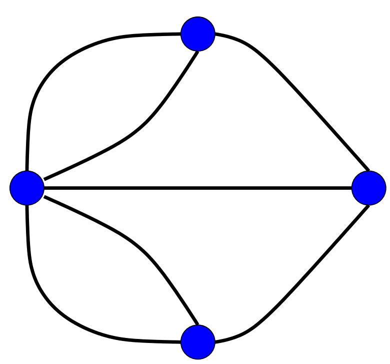
(b) Suppose that one of the bridges collapses, Can you now find a path across town that traverses each bridge exactly once?

(c) A new bridge is going to be built to replace the old one, but in a new location so that it will be possible to take a walk across town crossing each bridge exactly once. Where should such bridge be located?



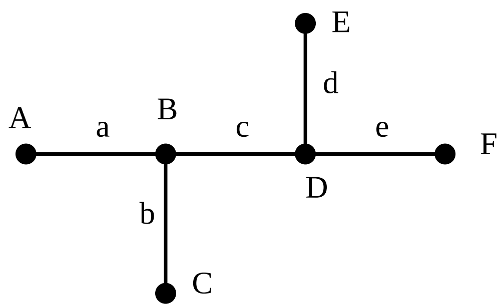


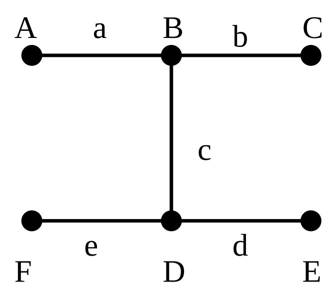
¶ **2.** Euler became intrigued by the problem, worked on it and solved it, in the following manner. He made a diagram out of the map of Königsberg, whereby each landmass was represented by a vertex, and each bridge was represented by an edge. Such diagram is called a graph or network.



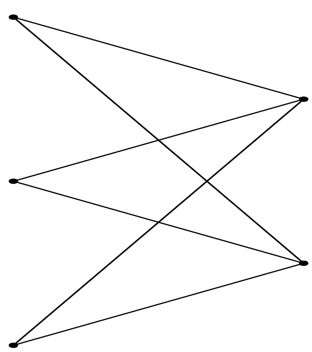
¶ 3. A *graph* is a mathematical object consisting of a collection of points $A, B, C, \ldots$ called **vertices** or **nodes**, and a collection of arcs $a, b, c, \ldots$ called **edges**. Each edge connects two distinct vertices, or else it connects a vertex to itself. In the latter case, such edge is called a **loop**.

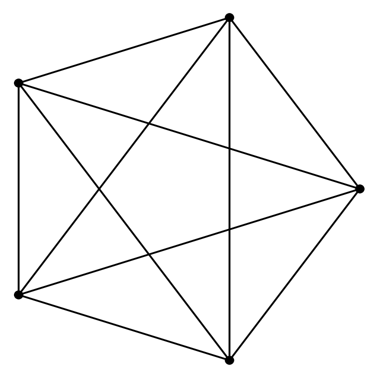
¶ 4. We usually represent a graph by drawing a collection of points (the vertices) and a collection of arcs between some pairs of such vertices. It should be noted that different drawings may in fact represent the same graph. like these two below:





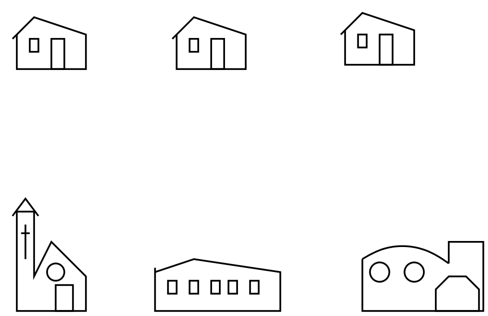
¶ 5. A little experimentation will show that many graphs can be drawn on a piece of paper (in the plane) in such a way that edges intersect only at the vertices. A graph that can be drawn in such manner is called a **planar graph**. Show that the graphs on the left below is a planar graph. What about the graph on the right?



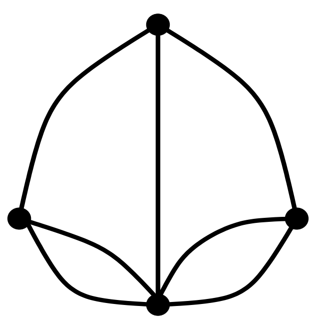


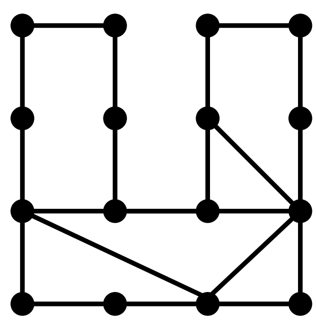
¶ 6. Many problems in recreational mathematics are based on the fact that many graphs are nonplanar. Perhaps the best know of all is the following, know as the three utilities problem:

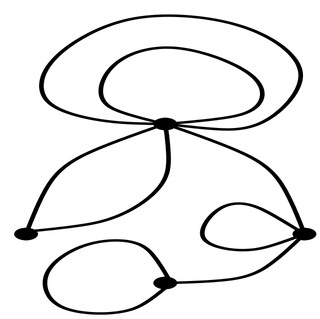
There are three houses in a county, and there is a church, a school, and a supermarket. The owners of the houses want to build roads from their properties to each the church, school, and supermarket, and want to do that in a way as to avoid crossings. Is that possible?



¶ 7. A concept that we need to introduce in order to solve the Königsberg bridge problem is that of **degree of a vertex** v. This is the number of edges that are incident to that vertex v, loops at v being counted twice because they have both ends at v. If the degree of v is odd, then v is called an off vertex, otherwise it is called an even vertex. The **total degree** of a graph is the sum of the degrees of all its vertices.
For the graphs below, determine the degrees of their vertices, and then their total degree:



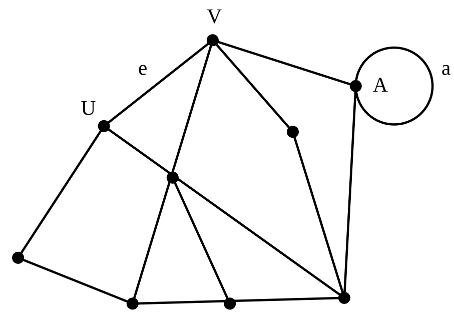




¶ 8. (a) Explain why the total degree of a graph is always an even number. (Hint. What is the relation of this number to the number of edges of the graph?)

(b) Explain why there is always an even number of vertices of odd degree.

¶ **9.** Other terminology that is used in graph theory and that will be relevant to solving the Königsberg Bridge Problem, is the following. A **path** in a graph is a sequence of nodes and edges (not necessarily distinct) of the form $AaBbCcD\ldots zZ$ so that any consecutive node-edge-node UeV is such that e is an edge joining U and V, as UeV or AaA in this figure:



A path of the form $AaBbC\ldots zZ$ is said to start at A and end at Z. A path that starts and ends at the same vertex is called a **circuit**. A circuit is always a path, but not conversely.
Two nodes A and B are joined by a path if there is a path that starts at A and ends at B. A graph with the property that any two of its nodes can be joined by a path is said to be **connected**.
The **length** of a path is the number of edges that compose it.

¶ **10.** A path that traverses every edge of a graph exactly once is called an **Euler path**. An **Euler circuit** is defined analogously: it is a circuit that traverses every edge of the graph exactly once. Every Euler circuit is an Euler path, but not conversely.

¶ **11.** If a graph has an Euler circuit, then the degree of each of its vertices must be even: you arrive at a vertex via an edge a and you leave that vertex via a different edge $b \neq a$. So each time that your Euler circuit passes through a vertex, it consumes 2 edges form the degree of that vertex.

Theorem 1. *A graph with an odd vertex cannot have an Euler circuit.*

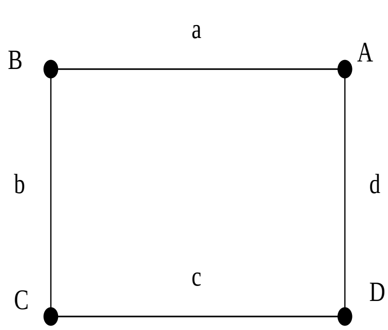
Nevertheless, a graph with an odd vertex could have an Euler path. But the same reasoning based on parity proves the following.

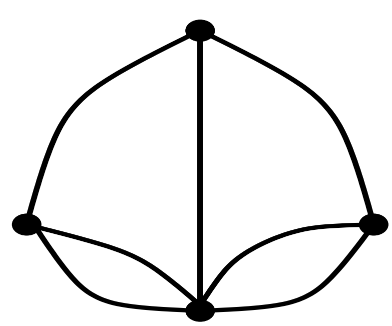
Theorem 2. (a) *If a graph containing an odd vertex V has an Euler path, then such path must begin or end at V.*

(b) *If a graph containing two odd vertices V and W has an Euler path, then such path must begin at V and end at W, or viceversa.*

(c) *A graph with more than two odd vertices cannot have an Euler path.*

¶ **12.** The graph on the left has an Euler circuit. Does the graph of the bridges of Königsberg have an Euler circuit? an Euler path?



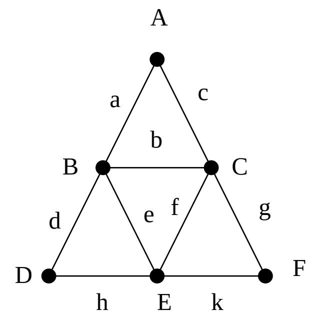


¶ **13.** Euler gave in fact a necessary and sufficient condition for the existence of an Euler path or an Euler circuit.

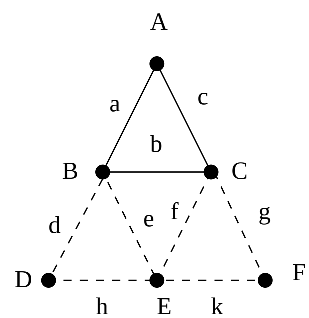
Theorem 3 (Euler). (a) *A graph has an Euler circuit (or cycle) if and only if it is connected and all its vertices have even degree.*

(b) *A graph has an Euler path if and only if it is connected and it has at most two vertices of odd degree.*

His proof of this theorem actually gives an algorithm for constructing an Euler path or an Euler circuit. We will explain this algorithm for the this graph:

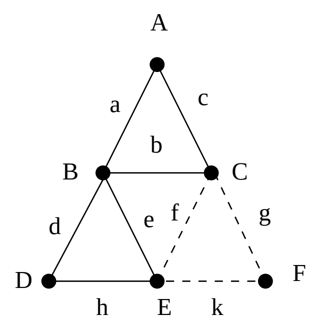


(a) Start a path at any vertex, say A, and go as far as you can without repeating edges, until you can no longer continue. This will be because you must have arrived at A and you have no edges left in A to continue. Thus you have traced a circuit Γ with no repeated edges.



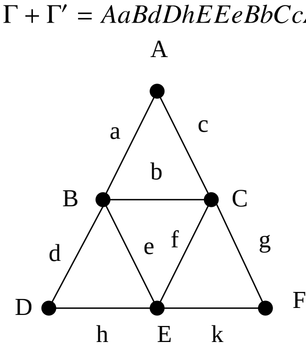
$\Gamma = AaBbCcA$

(b) If that circuit Γ is not an Euler circuit, there must be some edges in the graph which are not traversed by it. And because the graph is connected, there must be some vertex in Γ with unused edges Choose one of them, say B, and construct a path with no repeated edges that goes as far as possible, using edges not contained in Γ. Eventually you will get stuck building that path, and that would be because you have arrived at the vertex B and there are not unused edges at B. Thus you have constructed a circuit Γ' through B with no repeated edges.



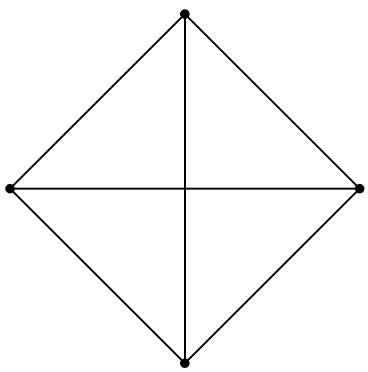
$\Gamma' = BdDhEEeB$
$\Gamma + \Gamma' = AaBdDhEEeBbCcA$

(c) Insert this circuit Γ' into Γ to obtain a longer circuit $\Gamma + \Gamma'$ with no repeated edges. If there are no unused edges left in the graph, then $\Gamma + \Gamma'$ must be an Euler circuit. If there are edges left, go to back to (b) and repeat the precess: the circuit $\Gamma + \Gamma'$ has some vertex without unused edges. Pick one of them, say C and construct a path with no repeated edges nor edges from $\Gamma + \Gamma'$ that goes as far as possible. Eventually you will arrive back at C and have obtained another circuit Γ''. Insert this into $\Gamma + \Gamma'$ to create a longer circuit with no repeated edges $\Gamma + \Gamma' + \Gamma''$, and so on.

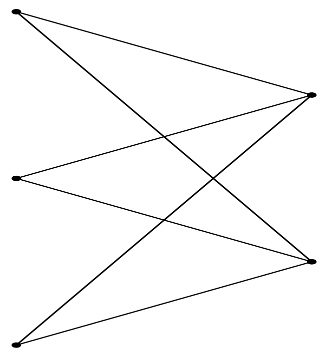


$\Gamma'' = CgFkEfC$
$\Gamma + \Gamma' + \Gamma'' = AaBdDhEEeBbCgFkEfCcA$

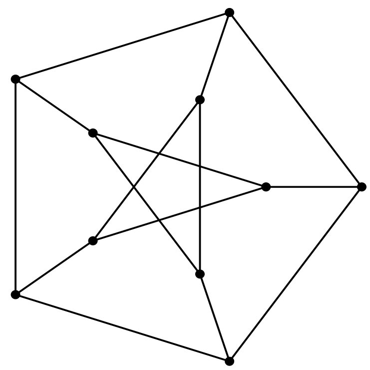
¶ **14.** Determine which of the following has an Euler path, an Euler circuit, or neither.



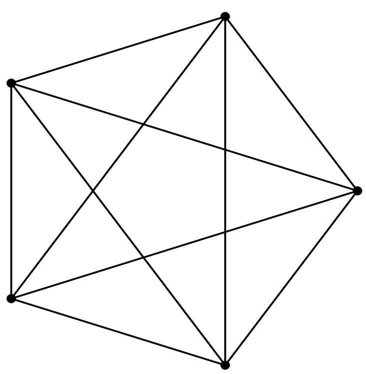
(a) CompleteGraph[4]



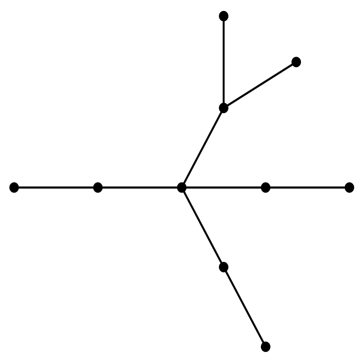
(b) CompleteGraphs[3,2]



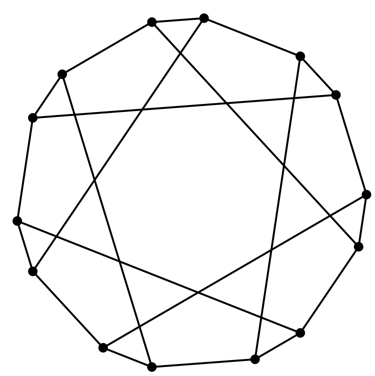
(c) PetersenGraph



(d) CompleteGraph[5]



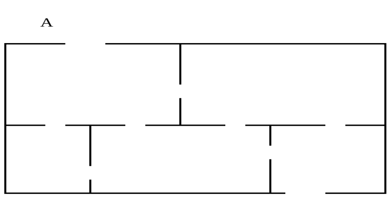
(e) RandomTree[10]



(f) CageGraph[3,6]

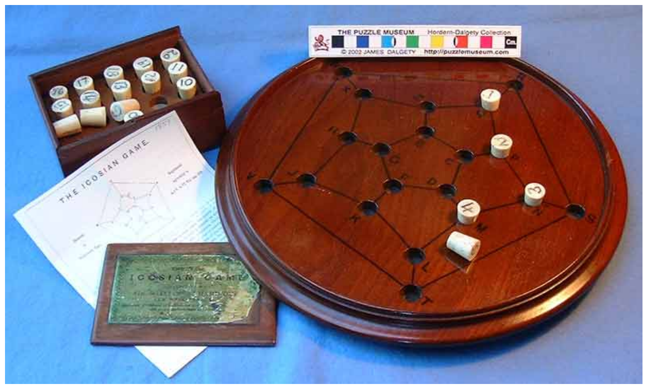
¶ **15.** Here are two other problems that reduce to graphs problems that you can solve using Euler's method.

(a) Below is the floor plan of a house. Is it possible to enter through the door at A, travel through all the house passing through each doorway exactly once, and exit through the door at B?

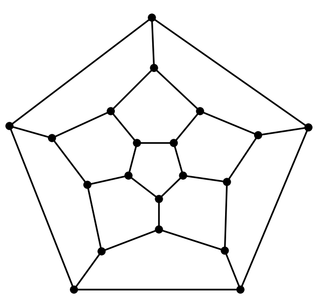


(b) Three missionaries were conducting three natives to the mission school. On the way they came to a crocodile infested river that had to be crossed. To do so, they only had available a canoe that could carry only two people, but only one of the missionaries and one of the natives could be trusted to paddle without tripping it. Also, because the natives were also ferocious cannibals, the missionaries did not want to create a situation in which they were outnumbered on either shore. How could they cross the river?

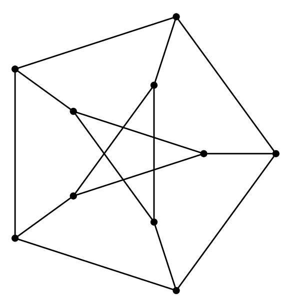
¶ **16.** A concept related to that of Euler path (or circuit) is that of Hamiltonian path (or circuit). A path in a graph is called a **Hamiltonian path** if it visits each vertex exactly once. A circuit in a graph is a Hamiltonian circuit if it visits every vertex exactly once, except for the first and last vertices, which are necessarily the same.
While similar to the problem of Euler paths, the problem of determining which graphs have Hamiltonian paths is still unsolved.
Hamiltonian paths and cycles are named after William Rowan Hamilton who invented the Icosian Game, which involves finding a Hamiltonian cycle in the edge graph of the dodecahedron.



¶ **17.** Find, if possible, a Hamiltonian path and a Hamiltonian circuit in the dodecahedral graph and in the Petersen graph pictured below.



Dodecahedral Graph



Petersen Graph

¶ **18.** Many graphs do not have Hamiltonian circuits. For example, construct a graph with $m + n$ vertices as follows. Place m vertices in a vertical column, and place the other n vertices on a parallel vertical column. Then put an edge joining each vertex of the first column with each vertex of the second column ($m = 3$ and $n = 2$ in the figure). If m is bigger than n, this graph does not have a Hamiltonian cycle. Can you explain why?