¶ **1** (Euler's Formula)**.** For each of the regular polyhedra we note that the alternating sum

$$F - E + V = 2$$

This identity, known as Euler's formula, holds true for all simple polyhedra, not just regular ones.

(a) Draw a pyramid with base any polygon and verify that Euler's formula holds true.

(b) Draw a prism with base any polygon and verify that Euler's formula holds true.

¶ **2.** An antiprism is a polyhedron that has congruent parallel polygons for base and top, but rotated with respect to each other so that the top polygon has vertices lying directly over the midpoints of the sides of the base polygon, joined by a band of triangles around the sides.

(a) Draw an antiprism with squares for top and base, and equilateral triangles for all the side faces.

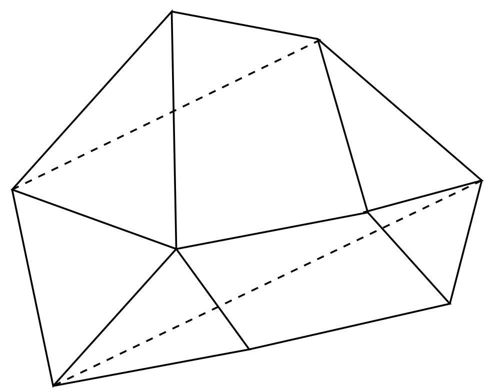
(b) If an antiprism has a base with n sides, find the number F of faces, the number E of edges, and the number V of vertices.

¶ **3.** To verify Euler's formula we proceed as follows.

(a) Suppose that your original simple polyhedron has F faces, E edges, and V vertices.

(b) Remove one of the faces, leaving behind its edges and vertices. The remaining surface can be "ironed-out" to a region in the plane in such a way that all its faces remain polygons bounded by straight lines. Such polygonal figure is called a planar diagram of the simple polyhedron. (These polygons will not in general be congruent to the original polygons of the polyhedron.)

(c) Draw a planar diagram for the following polyhedron:



¶ **4.** Your polyhedron and any of its planar diagrams have the same number of edges and vertices, but the planar diagram has one less face. We know perform a sequence of transformations to simplify the planar diagram but without changing the value of $F - E + V$.

(a) If your planar diagram has a face with more than three sides, we trace a diagonal on that side. With transformation we add one edge and one face, but we do not change the number of vertices. Thus $F - E + V$ remains unchanged. We continue this process until all the faces are triangles.

(b) This new planar diagram is composed of triangles. Some of this triangles have edge(s) on the sides of the diagram, other are completely inside.

(c) If ABC is a triangle with just one edge, say AC, on the boundary, we remove from the diagram that part of the triangle that does not belong to some other triangle. That is, we remove the inside of ABC, the edge AC, and leave behind edges AB and BC, and vertices A, B, and C.

We have removed 1 face, 1 edge, and no vertices and therefore the value $F - E + V$ remains unchanged.

(d) If a triangle PQR has two edges, say PQ and PR, on the boundary of the diagram, then we remove the face PQR, the edges PQ and PR, and the vertex P, leaving behind the edge QR and the vertices Q and R. Thus we removed 1 face, 2 edges, and 1 vertex, and so $F - E + V$ remains unchanged.

(e) If the triangle UVW is has all three sides on the boundary of the diagram by a single vertex, say W, but the diagram consists of more than just one triangle, we remove the triangle UVW, the edges UV, UW and VW, and the vertices U and V. Thus we removed 1 face, 3 edges, and 2 vertices, and so $F - E + V$ remains unchanged.

(f) By a sequence of transformations like those just described, we reduce our original planar diagram to a single triangle without changing the value of $F - E + V$. Since a triangle has $F - E + V = 1$, the original planar diagram has $F - V + E = 1$. Therefore, the original polyhedron has $F - E + V = 2$.

¶ **5.** We use Euler's formula $F - E + V = 2$ for the surface of the sphere to prove that there are only five regular polyhedra. Suppose that a regular polyhedron has F faces, each of which is a regular polygon with p sides, and that exactly q faces meet at every vertex. For example, for the cube we have $F = 6$ faces, each is a square (so $p = 4$) and $q = 3$ squares meet at each vertex.

(a) If E is the number of edges, then $2E = pF$ because each each edge belongs to two faces, so it is counted twice in the product pF.

(b) If V is the number of vertices, then $2E = qV$ because each edge has two vertices.

(c) Conclude from (a) and (b) and Euler's formula that $\frac{2E}{p} + \frac{2E}{q} - E = 2$.

(d) Conclude from (c) that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E}$.

(e) Explain why we always have that $p \geq 3$ and $q \geq 3$ (Hint. A polygon has at least three sides, and at least three sides must meet at polygonal angle.)

(f) Explain why p and q cannot be both ≥ 4.

(g) Deduce that if $p = 3$, then $q = 3, 4, 5$, and that if $q = 3$, then $p = 3, 4, 5$.

(h) List all the possibilities for p and q.

p	q	E

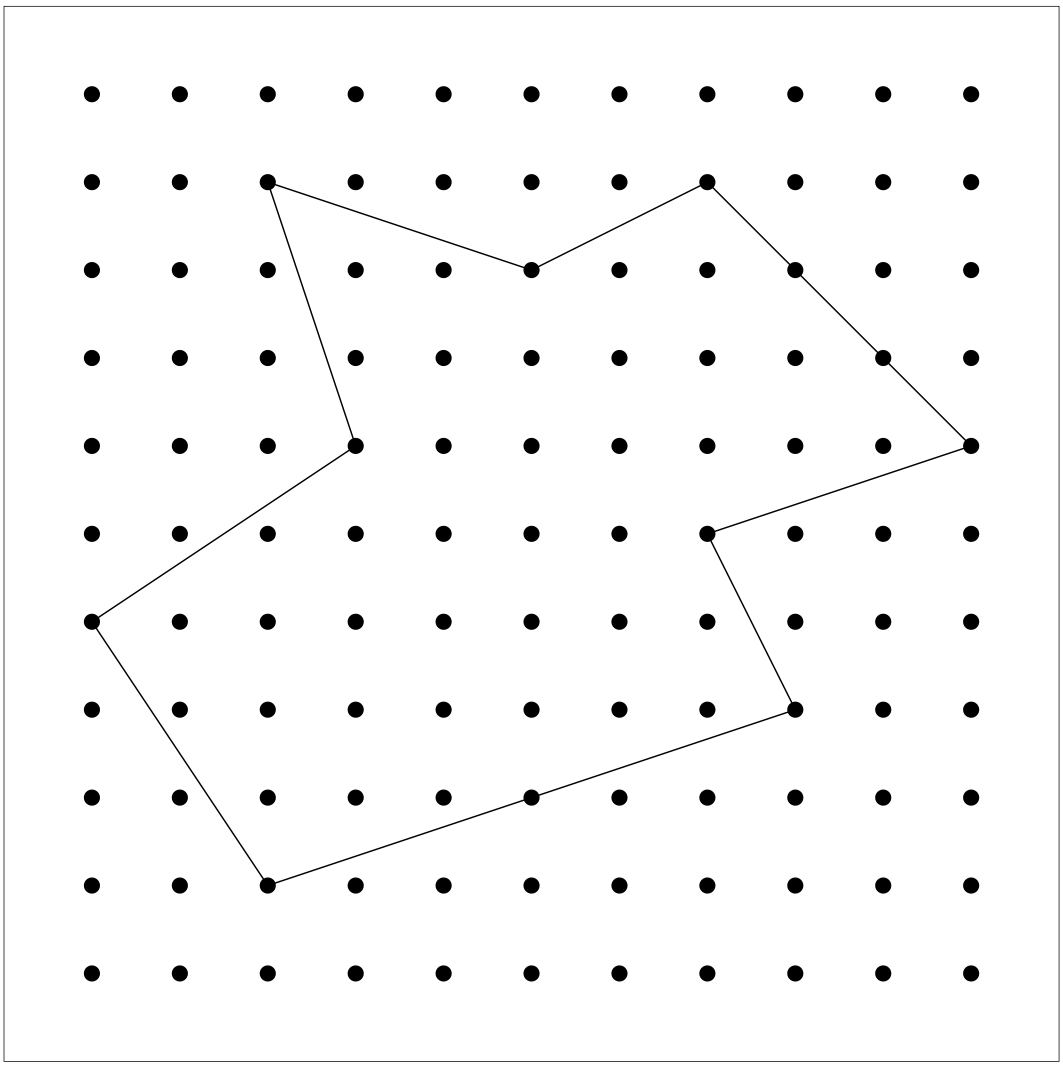
¶ **6** (Areas on the GeoBoard)**.** Points in the plane whose coordinates are both integers are called lattice points. Lattice points arise in a variety of problems. Here we apply Euler's formula to prove a surprising formula, discovered by G. Pick, for the area of a lattice polygon P (a polygon whose vertices are lattice points).

The formula is

$$\text{Area } P = V_I + \frac{1}{2} V_B - 1$$

where V_I is the number of lattice points inside the polygon P, and V_B is the number of lattice points on the boundary of P.

According to Pick's formula, what is the area of the lattice polygon in this picture?



¶ **7.** A lattice triangle that contains no lattice points in its interior is called a *simple* lattice triangle. Note that any lattice polygon can be divided into simple lattice triangles.

Divide the lattice polygon in the figure into the least possible number of simple lattice triangles.

¶ **8.** To verify Pick's formula we need the fact that the area of a simple lattice triangle is $\frac{1}{2}$.

Divide your lattice polygon into simple lattice triangles, Then cut out two copies of the polygon and glue them along their boundaries so as to build a simple polyhedron whose faces are triangles, twice as many as simple triangle the original lattice polygon had. Let V, E, F be the vertices, edges, and faces of this polyhedron, so that $F - E + V = 2$, according to Euler's formula.

(a) Explain why we have $V = 2V_I + V_B$

(b) Explain why we have $F = 2T$, where T is the number of simple lattice triangles of the original lattice polygon.

(c) Explain why $2E = 6T$. (Hint. All $F = 2T$ triangles provide a total of $6T$ edges. But there are repeats, because each edge is common to exactly 2 triangles.)

(d) Use Euler's formula $V - E + F = 2$ for simple polyhedra to prove that $T = 2V_I + V_B - 2$.

(e) Use the fact that each simple lattice triangle has area $\frac{1}{2}$ to deduce Pick's formula.

¶ **9.** What is the area of the orchard in the figure below if the rows and columns of trees are 6 feet apart?