¶ **1.** A tiling (by polygons) of the plane is an arrangement of polygons fitting together to cover the plane without leaving any gaps or overlapping, and so that the tiles fit edge to edge exactly.

We have study regular tilings, in which each tile is congruent to a regular polygon, and we have studied semiregular tilings in which each tile is congruent to one of several regular polygons.

We were able to show that there is a total of 3 regular tilings and 8 semiregular tilings.

¶ **2.** If we consider tilings with all tiles congruent to a single convex polygon but not necessarily a regular one, then the possibilities are numerous and the problem becomes truly interesting.

It can be shown that no convex polygon with more than six sides can tile the plane, so we will investigate only polygons with three, four, five and six sides.

¶ **3.** Any triangle can tile the plane. Here is how:

(a) Draw any triangle

(b) Create a congruent copy by rotating it 180 degrees about the middle point of one of its sides.

(c) Join the two copies into a parallelogram.

(d) Replicate this parallelogram via translations along directions parallel to its sides.

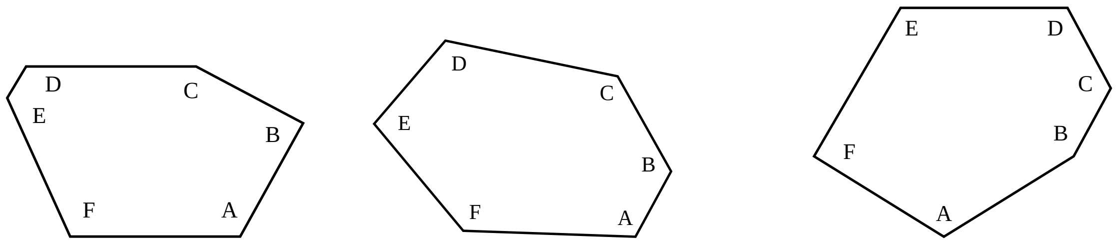
¶ **4.** It is almost obvious that any rectangle can be used to tile the plane. It is also easy, but nevertheless surprising, that any quadrilateral can be used to tile the plane. The idea is similar to that used in the case of a triangle.

(a) Rotate the quadrilateral about the midpoint of one of its sides to create a hexagon.

(b) The opposite sides of this hexagon are necessarily parallel because the angle sum of a quadrilateral is 360.

(c) Appropriate translations of this hexagon will cover the plane.

¶ **5.** The case of the hexagon was settled by Reinhardt in his thesis (1918). Any tiling by convex hexagons belongs to one of three classes: if you label the vertices as in the figure, then the conditions for each of the three types of tiling are:

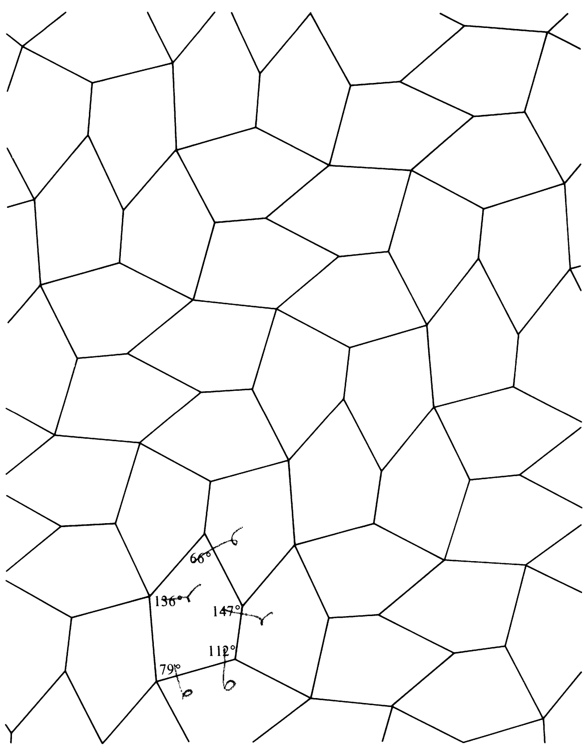


Angles $A + B + C = 360°$
Sides $AF = CD$

Angles $A + B + D = 360°$
Sides $AF = CD$ and $BC = DE$

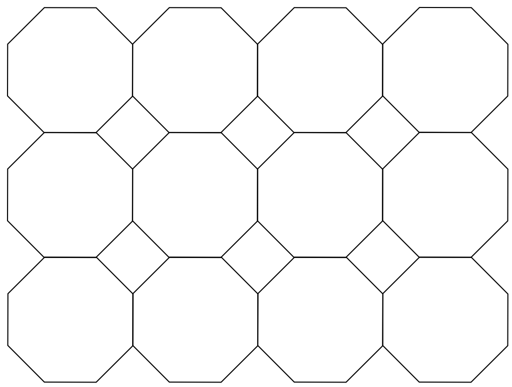
Angles $A = C = E = 120°$
Sides $AF = AB$, $BC = CD$, and $DE = EF$

¶ **6.** The case of tilings by copies of a convex pentagon proved to be much harder. Reinhardt found five types in his thesis, but three new types were discovered in 1967, and another one in 1969. The odyssey did not end here. In 1976, Marjorie Rice, a San Diego housewife whose only math background was that required in high school discovered two more types, and one more a year later, bringing the total pentagonal shapes to thirteen. One more was found in 1985. The achievements of Marjorie Rice are described in the paper by Doris Schattschneider in *The Mathematical Gardner*, D. E. Klarner (ed.) Prindle, Weber and Schmidt, 1981.

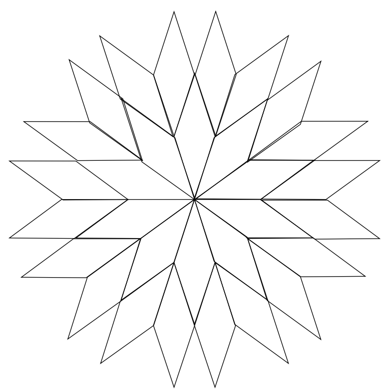


¶ **7.** All this tiling are of a type called *periodic*. That is to say, there is a region of the tiling (which may include several tiles or portions thereof) which can be used itself to tile the plane by simple translations. This region need not be unique, even if we take it to be of the smallest possible size.

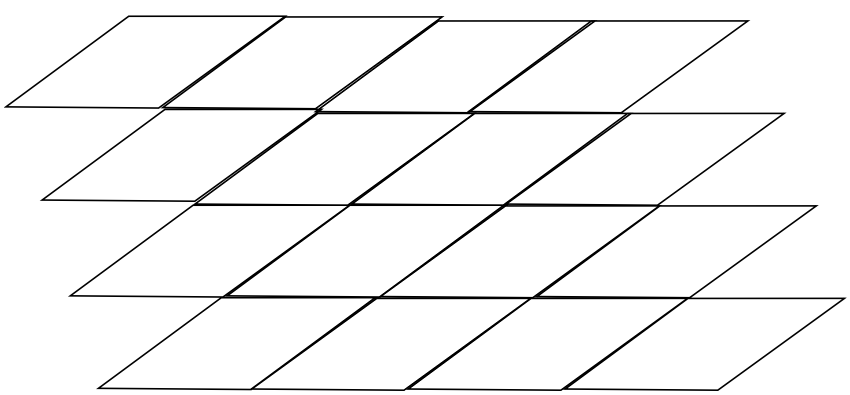
In the semiregular tiling by squares and octagons, find a smallest repeating region.



¶ **8.** The question arose as to whether all tilings are periodic. The answer is no: It if possible to form tilings of the plane with copies congruent to a single tile for which no translation preserve them. One such example is the following:

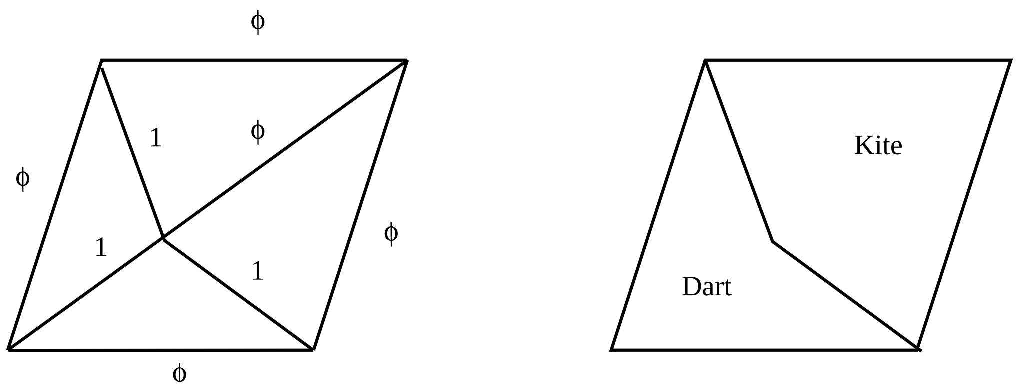


But this tile can be used to tile the plane periodically:



¶ **9.** The problem is thus to find a tile or set of tiles that tile the plane but do so in a non-periodic way: an aperiodic set of tiles. The problem was solved in 1964 by Berger who found an aperiodic set of over 20,00 tiles. Later he reduced that to a set of 104 tiles and another mathematician, Raphael Robinson, reduced it further to six tiles. A few years later, in 1974, Roger Penrose found an aperiodic set with only two tiles.

There are actually many versions of the so called Penrose tilings. One of the most popular is the *kite and dart*: The kite and dart are cut from a rhombus with side lengths equal to the golden ratio $\phi = \dfrac{1+\sqrt{5}}{2}$ and main diagonal equal to $\phi + 1$.



The two tiles can tile the plane in a periodic manner. Thus to obtain aperiodicity we make sure that the tiles are assembled according to the folllwoing two rules:

Rule I Edges should match edges of the same length.

Rule II Whenever two edges are paired, the signs (+ or −) at their endpoints should match.