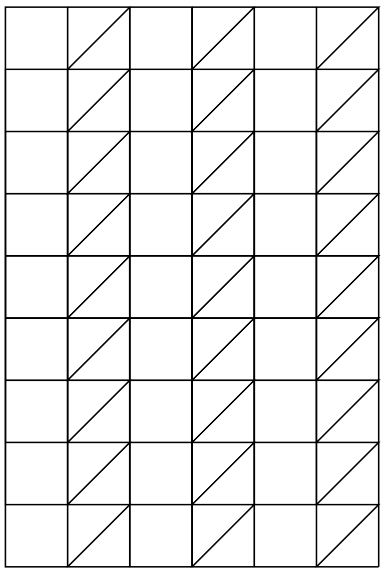
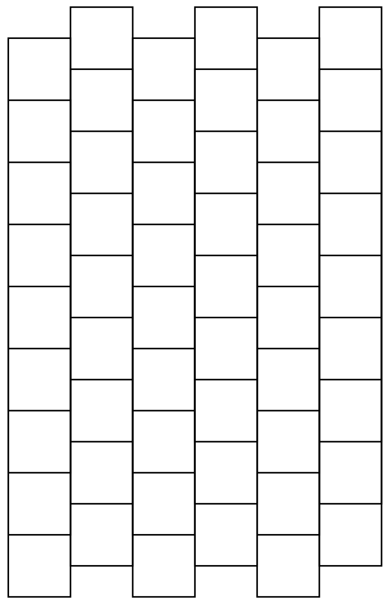
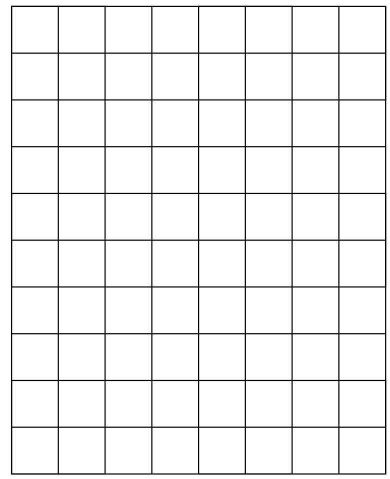
¶ **1.** A tiling (by polygons) of the plane is an arrangement of polygons fitting together to cover the plane without leaving any gaps or overlapping, and so that the tiles fit edge to edge exactly. A *regular tiling* is a tiling of the plane made out of polygons that are all congruent to a regular polygon.

Determine which of the following are regular tilings. Explain.

¶ **2.** It turns out that there are not that many regular tilings. For the problems that follow it will be useful to tabulate the vertex angles of a regular polygon

Polygon	Number of Sides	Vertex Angle
Triangle	3	
Square	4	
Pentagon	5	
Hexagon	6	
Heptagon	7	
Octagon	8	
Nonagon	9	
Decagon	10	

¶ **3.** You notice that the vertex angle of a regular polygon increases with the number of sides, according to the following relation. It is at least 60 degrees and at most 180 degrees (it is never exactly 180). The vertex angle, $V(n)$, of a regular n-sided polygon is given by is the function of n given by

$V(n) =$

¶ **4.** (a) In a tiling by polygons, the angles at a vertex must add up to how many degrees?.

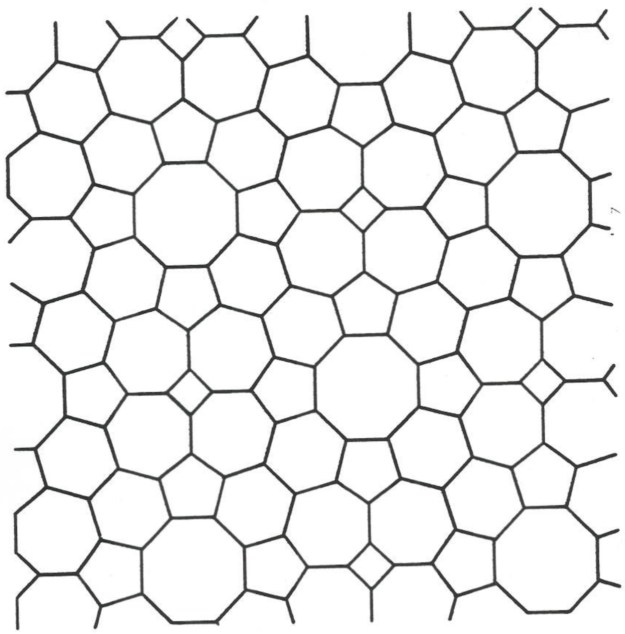
(b) In a regular tiling, what is the least number of polygons around each vertex?

(c) In a regular tiling, what is the largest number of regular polygons that can fit around a vertex?

¶ **5.** Which regular polygons can tile the plane?

¶ **6.** *Semiregular tilings* (or Archimedean tilings) are those tilings of the plane in which more that one regular polygon is employed, and each vertex has the same configuration.

¶ **7.** Can this be a section of a semiregular tiling?



!.1.6
'tiling by regular polygons" found in a children's
; book, *Altair Design* (Holiday [1970]).

¶ **8.** A regular octagon and a square can be used to construct an example of a semiregular tiling. Can you draw a sketch of this tiling? (Two octagons and a square meet at each vertex.)

¶ **9.** In a semiregular tiling, what is the least number of polygons around each vertex?

¶ **10.** In a semiregular tiling, what is the largest number of polygons around each vertex? (Look at the table of vertex angles; if there are many polygons, their vertex angles would be rather small.)

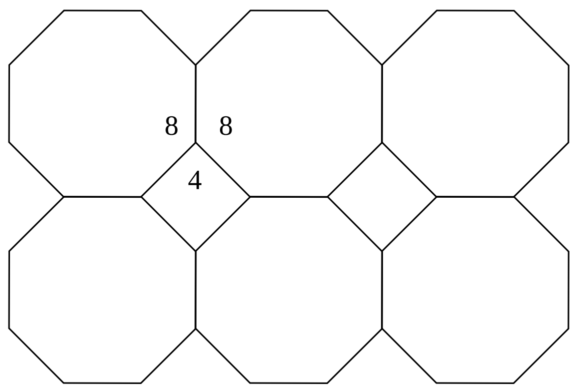
¶ **11.** (a) In a semiregular tiling, can you have four different polygons around a vertex?

(b) Therefore, if you have more than four polygons around a vertex, what can you say about them?

¶ **12.** Thus far we have obtained the following three rules that a regular or semiregular tiling must satisfy:

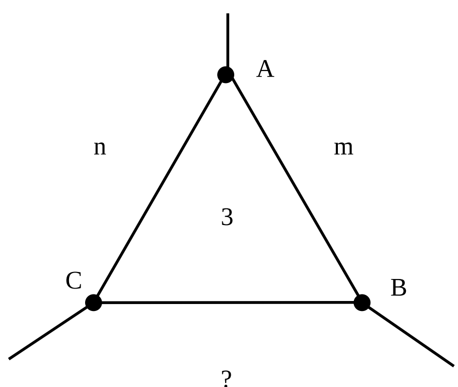
- **Rule 1** The angles of the polygons meeting at a vertex sum up to 360 degrees.
- **Rule 2** There are at least 3 polygons and no more than 6 polygons meeting at a vertex.
- **Rule 3** No semiregular tiling can have four different types of polygons meeting at a vertex.

A regular or semiregular tiling is determined by its vertex configuration. This is a string of whole numbers obtained by reading the number of sides of a polygon around a vertex (counterclockwise). Thus, the symbol `4.4.4.4` denotes that there are 4 squares around a vertex. The symbol `3.3.3.3.3.3` denotes that there are 6 equilateral triangles around a vertex. The symbol `4.8.8` denotes that there is one square and two octagons around each vertex. This is the same symbol as `8.4.8`, as we could start reading the labels at any polygon:



We will attempt to write down all possible symbols for the vertex configurations of semiregular tilings.

¶ **13.** Look at the figure below:

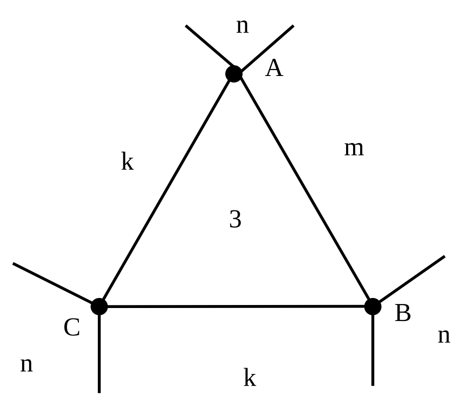


(a) From vertex A, the symbol is `3.m.n`, so "?" must be ...

(b) Note that a regular triangle, a regular heptagon, and a 42-sided regular polygon have vertex angles of 60 degrees, $128\frac{4}{7}$ and $131\frac{3}{7}$ respectively, which add to 360 degrees. So a vertex configuration `3.7.42` appear plausible but (a) shows that is not possible for a semiregular tiling.

¶ **14.** Is it possible to have a semiregular tiling with vertex configuration `5.m.n` with $m \neq n$? What about `k.m.n` with `k` odd and `m ≠ n`?

¶ **15.** Another vertex configuration that cannot occur for a semiregular tiling is `3.k.n.m` unless `k=m` Look at the figure below



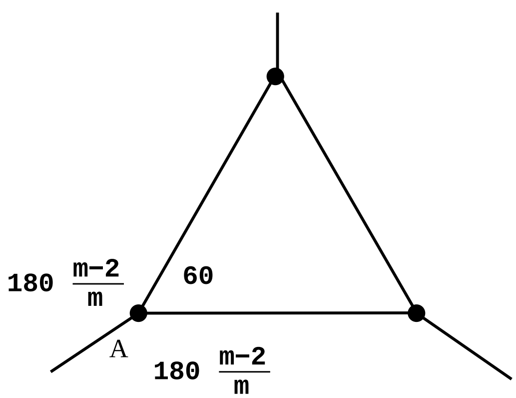
¶ **16.** We have obtained two additional rules for a regular or semiregular tiling:

- **Rule 4** No semiregular tiling can have a vertex configuration of the form `k.m.n` with `k` odd and `m≠n`.
- **Rule 5** No semiregular tiling can have a vertex configuration of the form `3.k.n.m` unless `k=m`.

¶ **17.** The five rules that we have found set some parameters for our search of regular an semiregular tiling.

Notice that there are some symbols for vertex configurations which, while appropriate because the angles at a vertex add up to 360, cannot be realized by a semiregular tiling. For example, show that `3.7.42` is a possible vertex configuration because the angles at the vertex sum to 360, but it does not occur in a semiregular tiling.

¶ **18.** By Rule 4, a vertex configuration with symbol `3.m.n` must have `m=n`. What are the possible values for `m`? Look at the angles at vertex *A* in this figure, then add the angles



Angle sum at vertex *A* is

$$60 + 180\frac{m-2}{m} + 180\frac{m-2}{m} = 360$$

What is m? (Solve for m.)

Try to fill out the table below, listing all possible vertex configurations for a regular or semiregular tiling:

Symbol	No. Polygons	Description of vertex
3.3.3.3.3.3	6	
3.3.3.4.4	5	
	5	
	5	
4.4.4.4	4	
	4	
	4	
	4	
4.8.8	3	
6.6.6	3	
3.7.42	3	
	3	
	3	
	3	
	3	
	3	
	3	
	3	

There is a total of 17 possible configurations, and 4 of them have each 2 possible arrangements giving a total of 21 possible vertex configurations. It turns out that some of this configurations do not extend to tilings of the plane.

Theorem 1. *There are exactly 3 regular tilings and 8 semiregular tilings of the plane.*

¶ **19.** If a vertex has symbol k.m.n, and one of the polygons is odd sided, say k is odd, then the other m=n. We can list the possibilities with a little bit of arithmetic.

(a) Write an equation showing that the angles at the vertex add up to 360 degrees.

(b) Use that equation to fill in the following table of possible solutions m and k

k	m

¶ **20.** A vertex symbol k.m.n with all even and $k \leq m \leq n$ can be analyzed similarly.

(a) If k = 4, write the angle equation linking m and n.

(b) Use that equation to fill in the following table of possible solutions m and k

m	n

(c) Do the same for k = 6.

¶ **21.** Can you draw sections of each of those tilings?