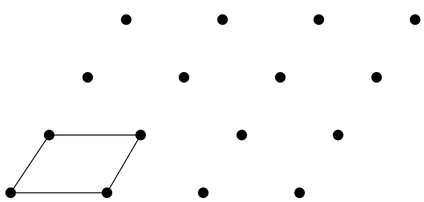
Math 311. Wallpaper Patterns 2
Name:

A Candel
CSUN Math

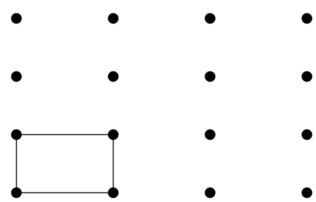
¶ **1.** Crystallographers have developed a notation for wallpaper groups. It is now well established and bellow we give a flow chart that allows to determine the name of a given wall paper group. The flowchart itself does not reveal all the features of a wallpaper pattern, and it is thus convenient to identify them here.

A wallpaper group always contains translations in two independent directions, and then perhaps other rigid motions, like rotations and glide reflections. This group of translations is called the lattice group of the wallpaper pattern: a lattice is obtained by taking a point of the pattern and all other points obtained from it by translations of the pattern. A parallelogram whose vertices are points of this lattice and contains no other points of the lattice inside it or on its sides is called a primitive cell. Notice that neither the lattice not the primitive cell are uniquely determined by the pattern. Sometimes the whole pattern can be easily obtained from this primitive cell. But sometimes a primitive cell is not easily seen, so the analysis of the pattern is better done by studying the rigid motions that preserve it rather than by studying a primitive cell. Usually, a primitive cell is chosen with centers of rotation of the highest orders as its vertices. There are two cases (when the primitive cell is a rhombus) in which a centered cell is chosen; this has size double that of the primitive cell.

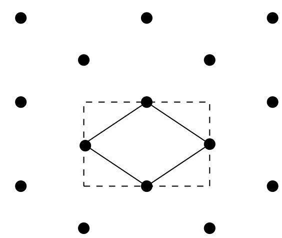
There are five kinds of primitive cells that arise: parallelogram, rectangular, square, rhombic, and hexagonal. The reason is that a pattern can only have rotational symmetry of orders 2, 3, 4 and 6. If there are no rotational symmetries but there are mirror lines and glide reflection lines, then the lattice will have parallel rows in perpendicular directions. Crystallographers choose primitive cells as pictured below; in the case of the rhombic cell, a centered cell is chosen which is made of two unit cells.



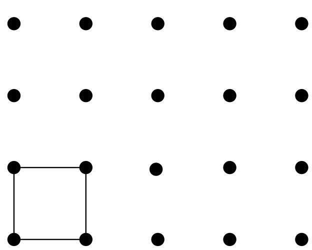
Parallelogram



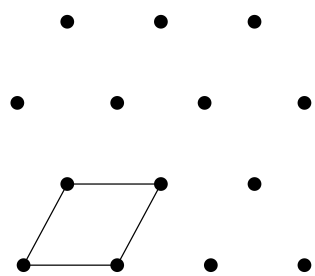
Rectangular



Rhombic



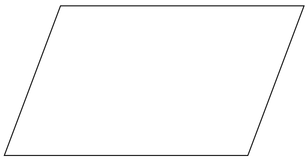
Square

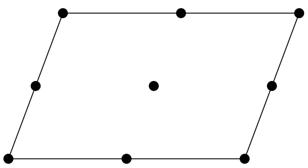


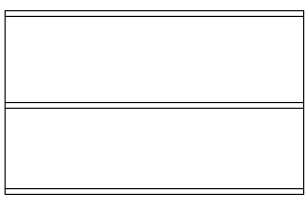
Hexagonal

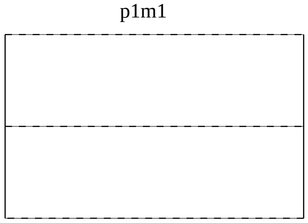
A primitive cell may overlap itself only along parts of its border when translated by the translation of the symmetry group of the pattern. However, if other symmetries are present (like rotations, and glide reflections), then some of those will take the primitive cell into positions that have regions in common with the original position. A minimal region withe the property that symmetries of the pattern take it to others that overlap along portions of its boundary, or not at all, is called a fundamental domain.

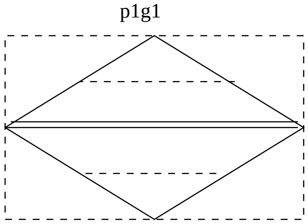
¶ **2.** Pictured below are primitive cells of the 17 wallpaper groups according to standard crystallographic notation. Each cell contains markers identifying the several symmetries of the pattern. A double line (==) indicates a mirror line. A dashed segment (- -) indicates an axis of a glide reflection. A point marked with a circle indicates a center of 180° rotation, a ■ indicates a center of 90° rotation, a ▲ indicates a center of 120° degree rotation, a hexagon indicates a center of 60° degree rotation.

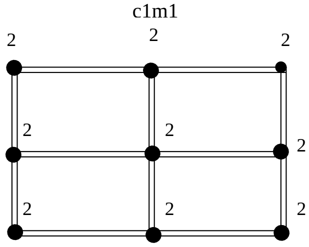

p111

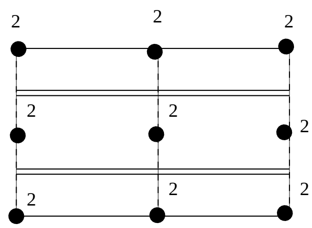

p211

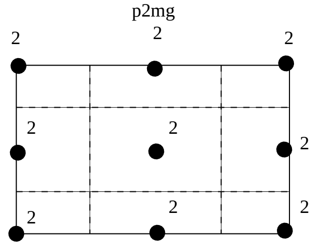

p1m1

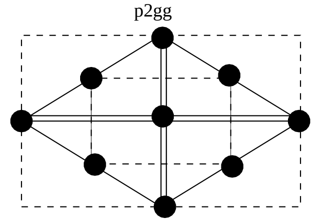

p1g1

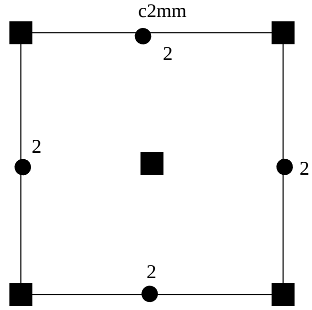

c1m1

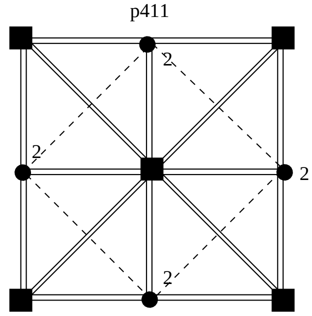

p2mm

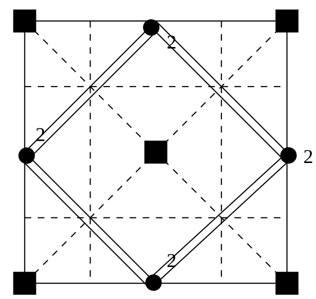

p2mg

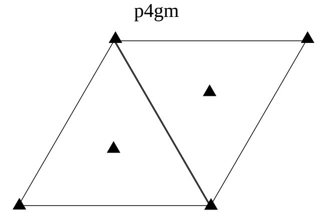

p2gg

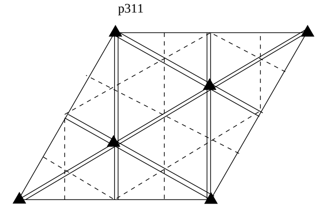

c2mm

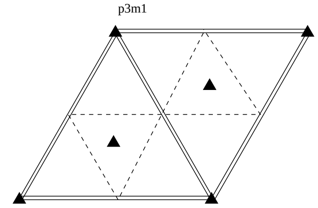

p411

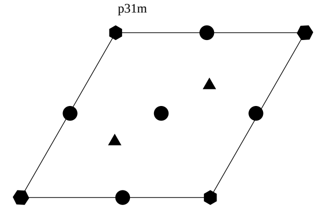

p4mm

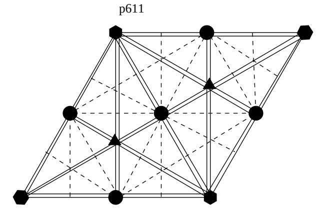

p4gm


p311


p3m1


p31m


p611


p6mm

¶ **3. Wallpaper patterns: Crystallographic Notation.** Crystallographers have standard notation for the 17 different wallpaper patterns. The full notation consists of 4 symbols:

c	1	m	m
p	2	g	g
	3	1	1
	4		
	6		

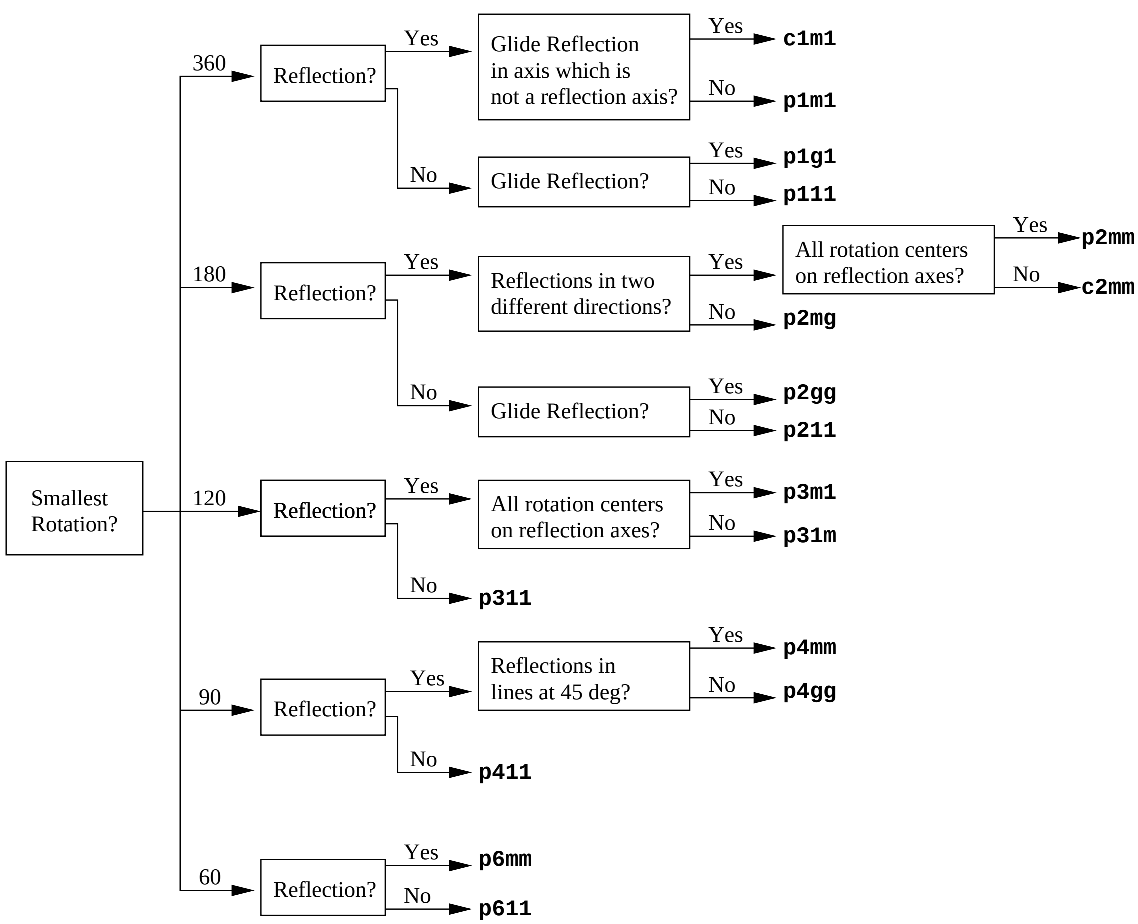
(a) The first symbol is either a `c` (for *centered cell*), or a `p` (for *primitive cell*) otherwise

(b) The second symbol indicates rotational symmetry. It is either of order `1`, `2`, `3`, `4`, or `6`, corresponding to 360, 180, 120, 90, or 60 degrees. The symbol is the largest applicable number: for example, if symmetries of 360, 120 and 60 degrees are present, then the second symbol is `6`.

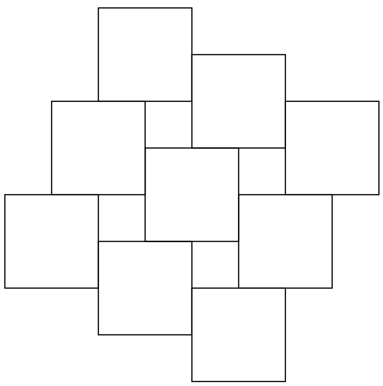
(c) The third symbol is either `m`, `g`, or `1`, corresponding to the presence of *mirror*, *glide*, or no reflection symmetry.

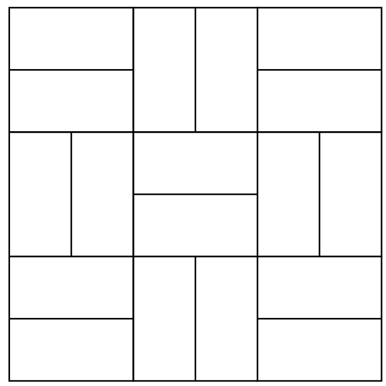
(d) The fourth symbol is either `m`, `g`, or `1`, and describes the symmetry relative to an axis at an angle to the symmetry axis of the third symbol. That angle would be determined by the second symbol: if that second symbol is `1` or `2`, then look for reflection or glide line at angle 180°; if the second symbol is `4`, then look for lines at angle 45°, and if the second symbol is `3` or `6`, then look for lines at angle 60°.

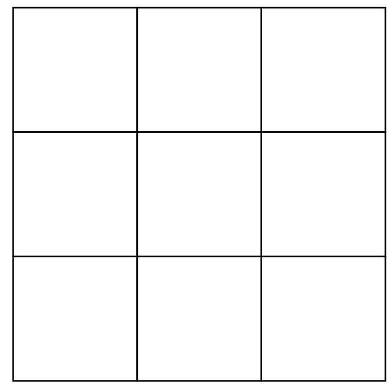
This notation is summarized in the following flowchart:



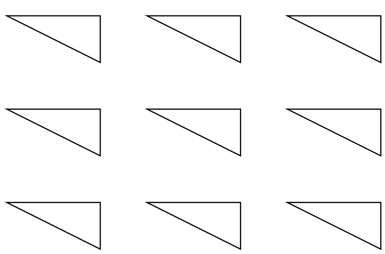
¶ **4.** Determine the crystallographic notation for the wallpaper groups of the following patterns. Then find a primitive cell and fundamental region for each.

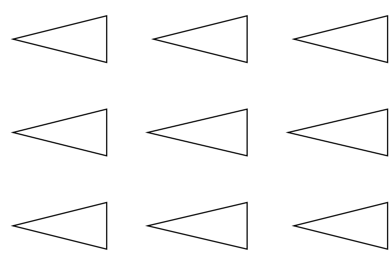


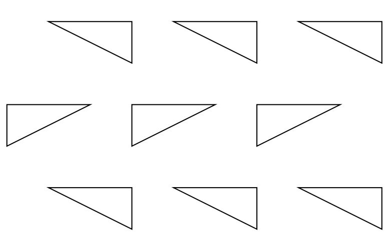


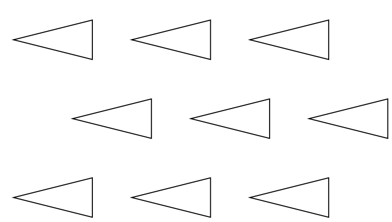


¶ **5.** Determine the crystallographic notation for the wallpaper groups of the following patterns. Then find a fundamental region for each.

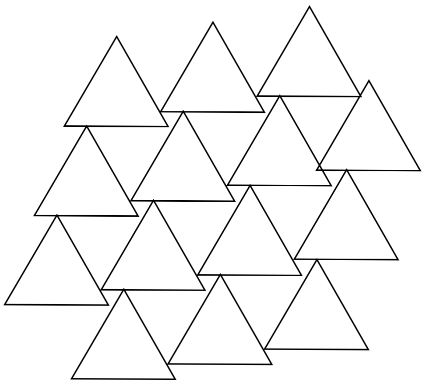
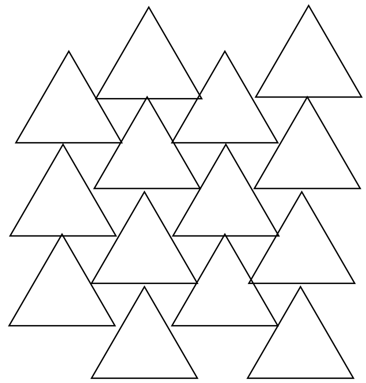
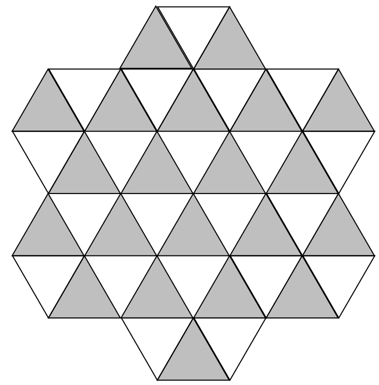




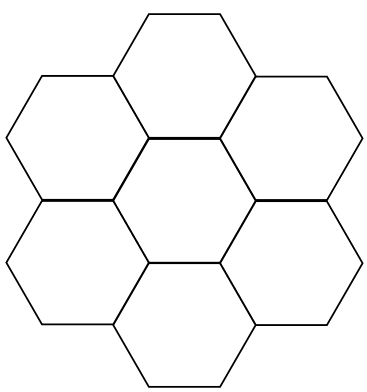
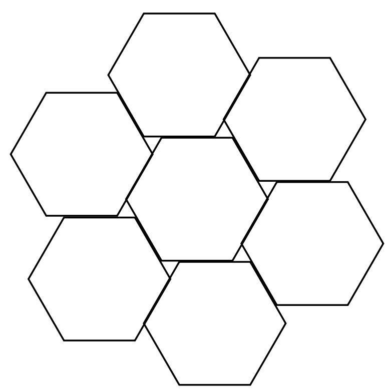




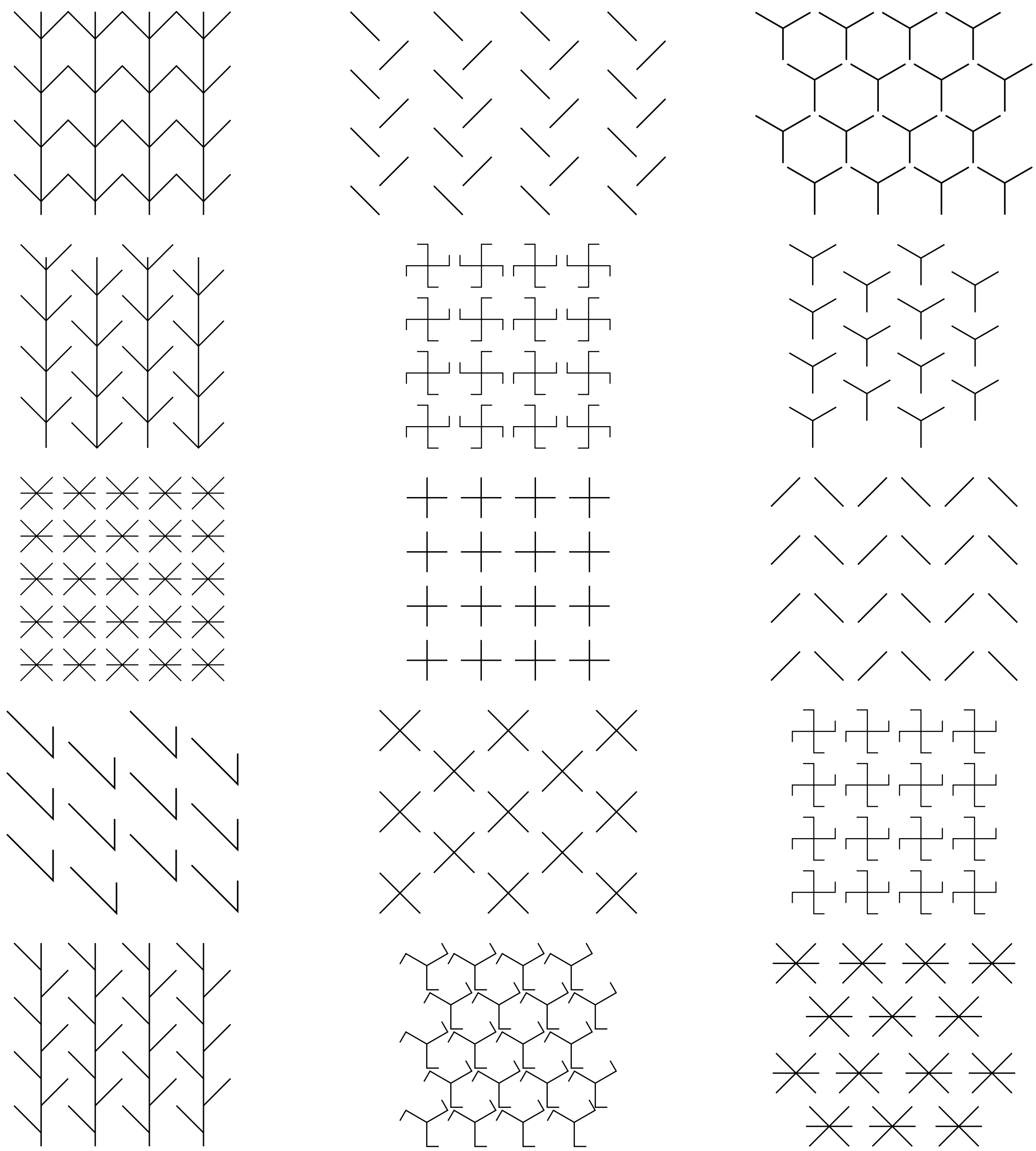
¶ **6.** Determine the crystallographic notation for the wallpaper groups of the following patterns. Then find a fundamental region for each.

¶ **7.** Determine the wallpaper group of the following patterns.

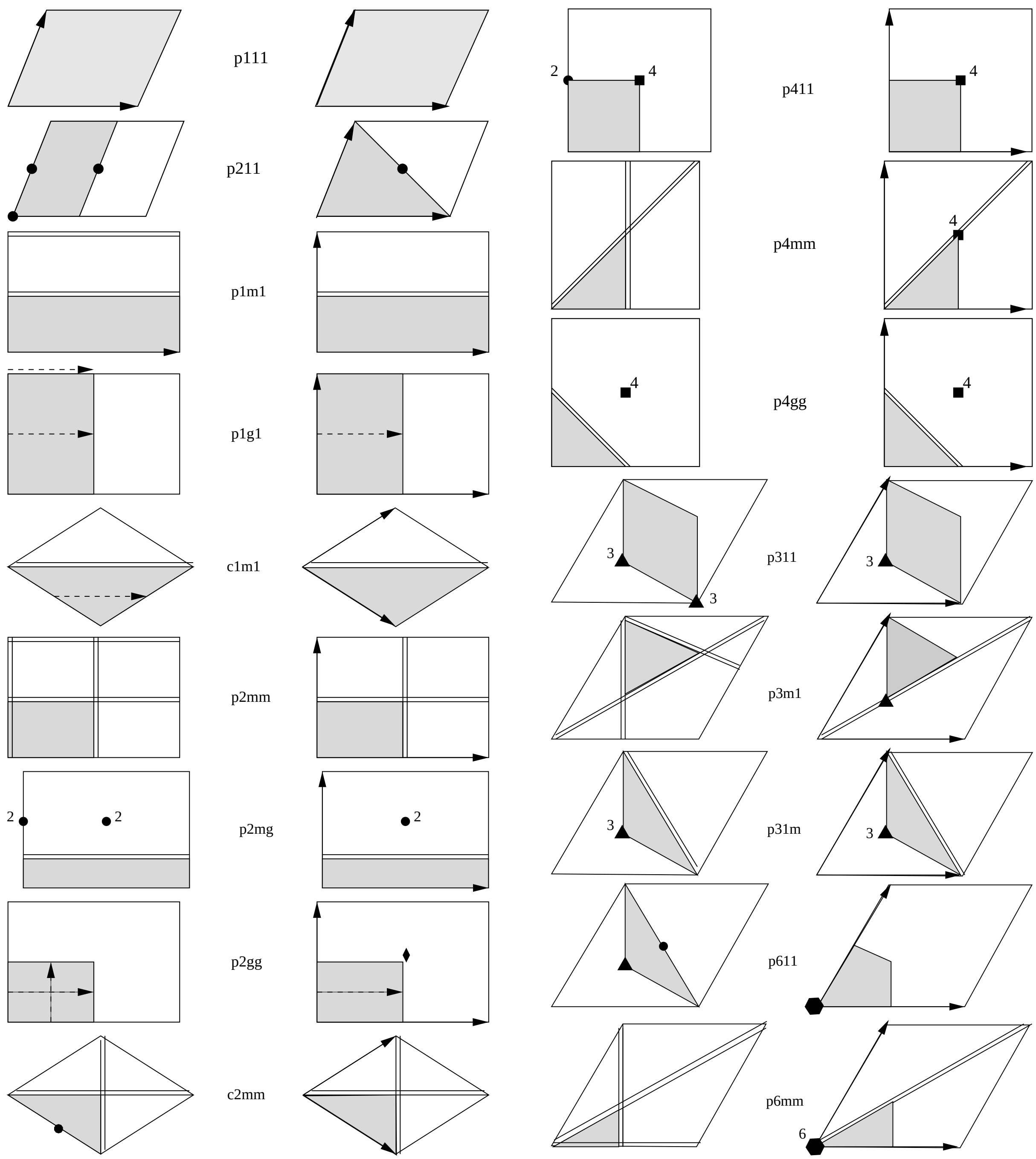



¶ **8.** Use the flowchart on the previous page to identify the crystallographic notation of the following wallpaper patterns.

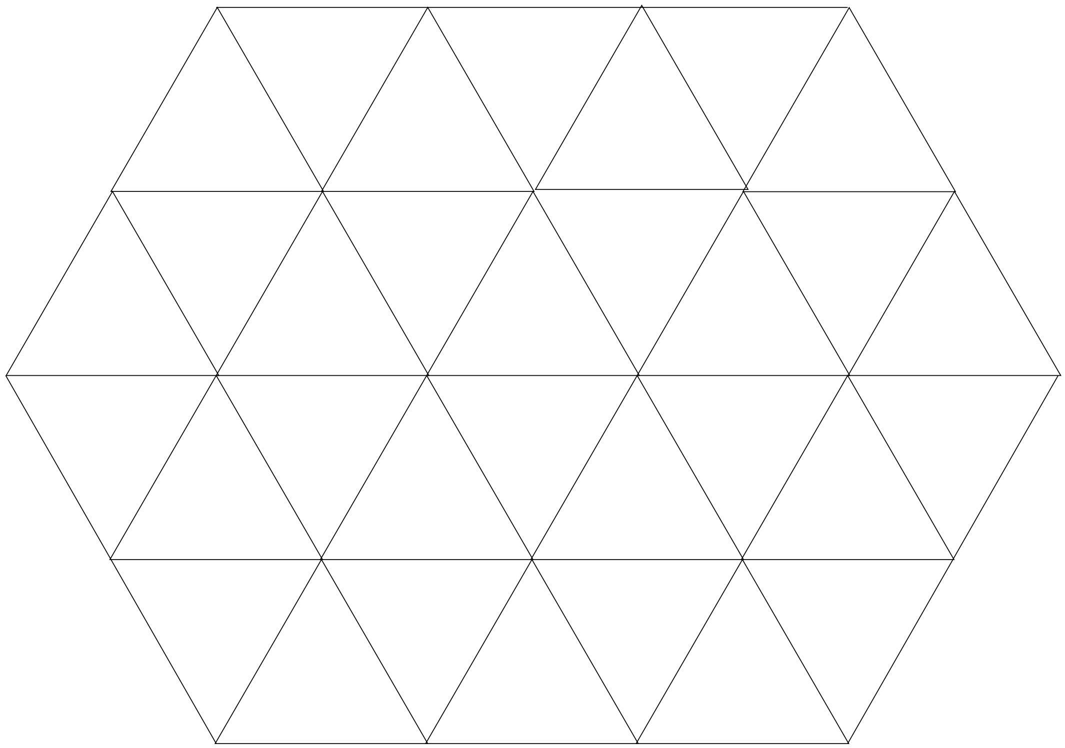


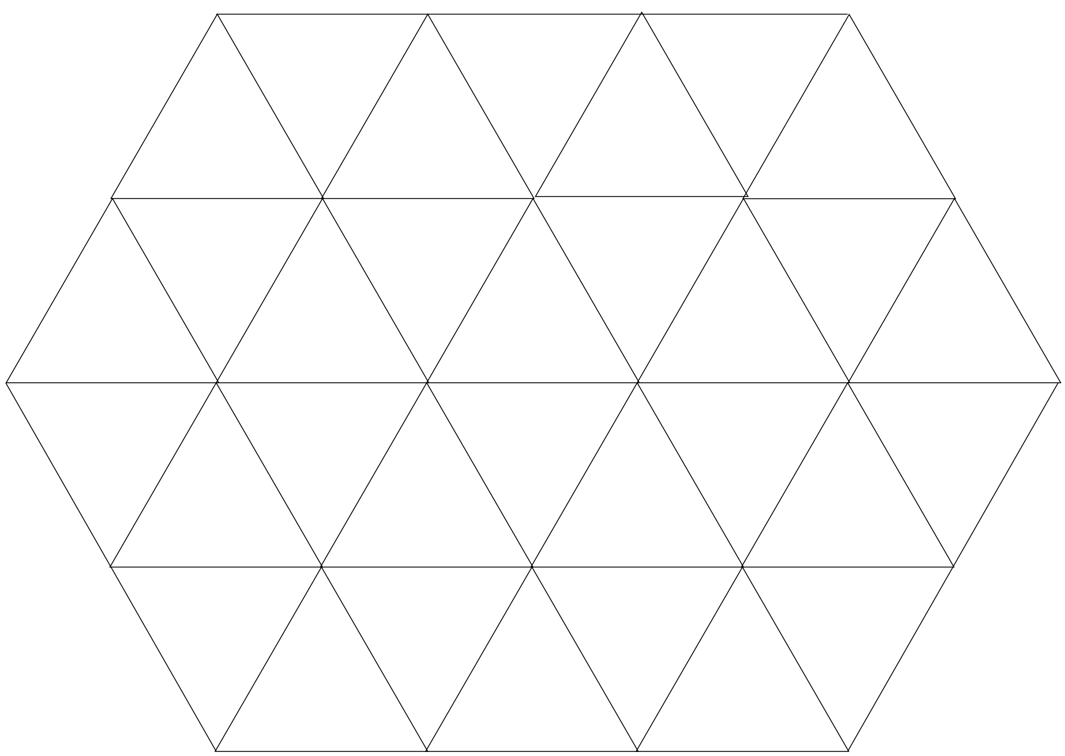
¶ **9.** Out of the 17 possible wallpaper patterns, there are 2 missing. Can you identify them and draw a section of each of them?

¶ **10.** The crystallographic description can be used to design wallpaper patterns with a given symmetry group. In the figure below, for each wallpaper group two diagrams are given. On the left figure, a primitive cell and a minimal set of generators is given, and a fundamental domain is shaded. On the right figure, a set of generators that includes the primitive cell unit translations is given.



¶ **11.** Use the figure in the previous Problem to designs patterns with symmetry group `p3m1` and `p31m`





¶ **12. M.C.Escher** (1898–1972) was a Dutch artist whose work illustrates in a dramatic way all the wallpaper patters. Use the crystallographic pattern flowchart to identify the following patterns from Escher's work.