¶ **1. Group theory** is the tool for the study of symmetry. A group is a collection of objects together with an operation for combining them. For example, one can have a group in which the objects are numbers and the combining operation is addition.

Not every set of operations on a collection of objects makes a group; the operations have to satisfy four properties. First, among the operations there must be one that leaves the system unchanged; this is the identity operation: just do nothing. Second, every object in the group must have an inverse, an "undo" action that returns the system to its former status. In the example of the mattress, every operation is its own inverse. Third, the group operations obey an associative law: if f, g, and h are three elements in the group we can combine them in two different ways $(fg)h$ or $f(gh)$, but the result is the same. Fourth, there is a closure requirement: combination of two elements f and g results in a new elements fg in the group.

A **group** is a collection of elements $\{A, B, C, \cdots\}$ and an operation $*$ between pairs of them such that the following properties holds:

(a) Closure.

(b) Identity element. There is a special element I such that $A * I = I * A$ for any A.

(c) Inverse. For any A there is an inverse A^{-1} such that $A * A^{-1} = A^{-1} * A = I$

(d) Associativity. The result of performing several operation is independent of the grouping, provided that the order is maintained: $A * B * C = (A * B) * C = A * (B * C)$.

¶ **2. Group of Permutations** A most pervasive example of a group is the group of permutations of a set of objects. For example, given the collection of numbers $\{1, 2, 3, 4, 5\}$, written in this order, a permutation of this set is the action of writing this set in a different order. To visualize the effect of a permutation on the set $\{1, 2, 3, 4, 5\}$ and make it easier to visualize the effect of combining permutations, we write them as follows. If the permutation consists in writing the numbers $1, 2, 3, 4, 5$ in the order $1, 3, 5, 4, 2$, then this permutation is visualized as $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix}$

(a) What is the effect of performing the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix}$ followed by the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}$

(b) What is the inverse permutation of $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}$

¶ **3. Rigid motions of the plane.** A rigid motion (isometry, congruence) of the plane is a transformation of the plane which leaves distances unchanged (you may know “rigid motion” as “congruence”). They preserve the size and shape of figures: any pair of points is the same distance apart after the motion as before. Any rigid motion of the plane must be one of the four types:

- Reflection (across a line).
- Rotation (around a point, the center of the rotation).
- Translation (in a particular direction).
- Glide reflection (across a line).

Performing one rigid motion after another results in another rigid motion, and thus must be of one of the types just described.

¶ **4.** The following properties of the symmetries of the plane are important.

(a) A **reflection** leaves a whole line of points fixed. This is the line of reflection (reflecting lines, axis of the reflection, mirror line). Points not on the line of reflection exchange sides under the reflection. A reflection changes orientation.

(b) A **rotation** has a single fixed point, the center of the rotation. All the other points move by the same angle (the angle of rotation) along circles with center at the center of the rotation.

(c) A **translation** moves all the points by the same amount along the same direction. There are no fixed points.

(d) A **glide reflection** is a combination of a reflection on a line followed by a translation in the same direction of the line of reflection. It does not have fixed points, but moves the line of reflection onto itself.

Reflections are the most fundamental examples of rigid motions because any other rigid motion can be obtained by combining at most three reflections.

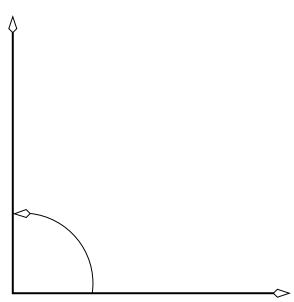
Combination of two reflections results in either a rotation or a translation. It is a rotation if the lines of reflection intersect. The point of intersection is the center of the rotation. The angle of rotation equals twice the angle from the first line of reflection to the second one.

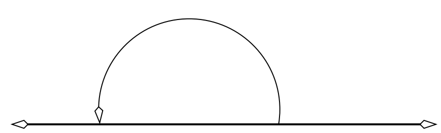
If the reflection lines do not intersect, that is, if they are parallel, the result of the combination is a translation in a direction perpendicular to the reflection lines by an amount equal to the distance between the lines.

Combination of three reflections results in what is called a glide reflection. This is a rigid motion that consists on a translation followed by a reflection on a line with the same direction as the translation.

¶ **5.** If you combine two rotations about the same center, the result is a rotation about the common center by an angle equal to the sum of the angles of each of the two rotations.

What is the effect of combining two rotations about different centers?


A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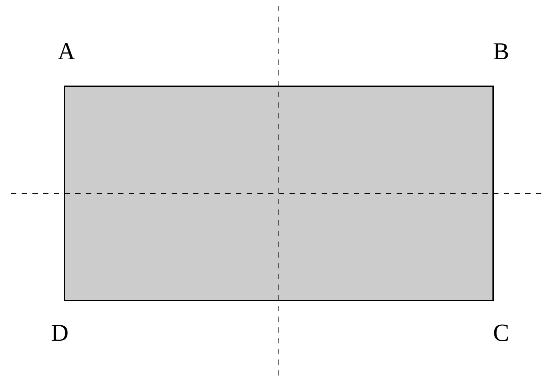
¶ **6. Symmetries of a rectangle.** Consider the rectangle in the figure below. Its symmetries are the rigid motions which bring it back to coincide with itself.



The symmetries of the rectangle are:

(a) ____________________

(b) ____________________

(c) ____________________

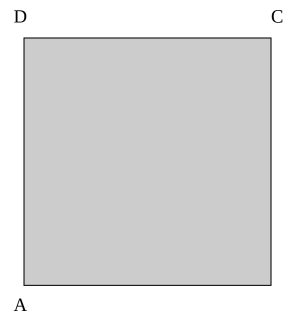
(d) ____________________

(e) ____________________

You should convince yourself that the isometries listed fulfill the four properties required for forming a group.

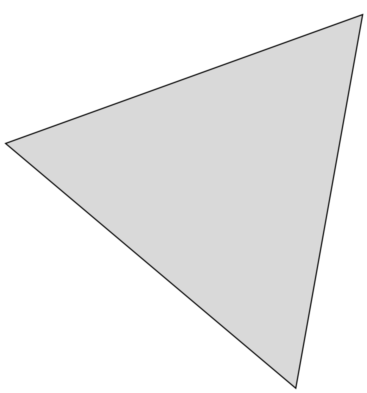
¶ **7.** Write down the group table for the symmetries of a rectangle.

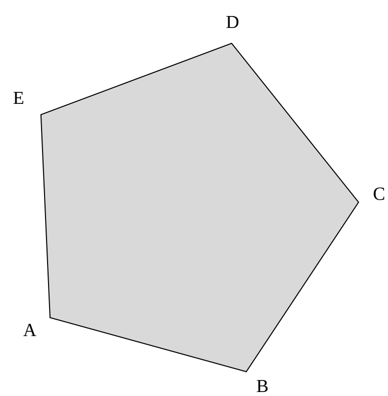
¶ 8. (a) Describe all the symmetries of the square. How many are reflections in lines? How many are rotations?



(b) One kind of symmetry transformation of the square is a reflection in the perpendicular bisector of two opposite sides. Another is a reflection in the line joining two opposite vertices (diagonal). Describe the symmetry transformation that results from doing a symmetry of the first type, followed by a symmetry transformation of the second type.

¶ 9. Identify all the symmetries of a regular polygon. Then write down the multiplication table for that group of symmetries. What permutations of the set of vertices of the polygon can be realized by symmetries of the regular polygon?

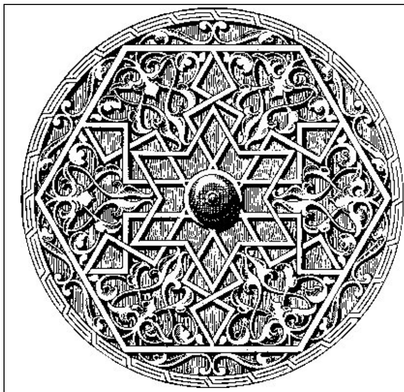
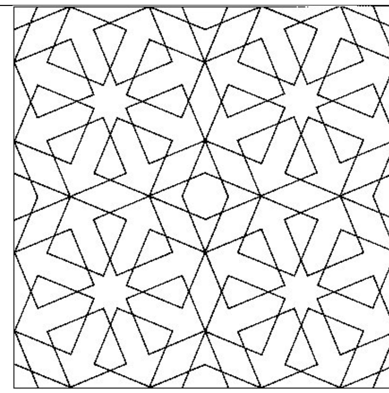




¶ 10. Symmetries of patterns

Given a pattern, we analyze it by determining which rigid motions preserve the pattern. These are referred as the symmetries of the pattern, and form a group. Pattern in the plane are usually divided into three groups:

- those that repeat in no direction (rosette patterns).
- those that repeat in exactly one direction (frieze patterns)
- those that repeat in more than one direction (wallpaper patterns)

		
(Rosette)	(Frieze)	(Wallpaper)

¶ 11. Mirror Symmetry If a figure can be divided by a line so that the part on one side of that line is the mirror image of the part of the figure on the other side then we say that the figure has mirror symmetry, and the line marking the division is called the mirror line (or reflection line, or line of symmetry).

Many letters in the alphabet have one or more mirror lines. (Sometimes that depends on the font.) Many numerals (both Arabic or roman) have mirror lines. Identify the mirror lines for each of the letters or words below.

A	★	WAVYTUMMUTYVAW
B	✻	XIXC
H	bdbdbd	❊
pId	✼	MOON
bId	✾	ABRACADABRA
➽		WOW
✖	dbpqdb	31

¶ 12. Rotational Symmetry Some figures remain unchanged when rotated about their center by certain angle; they have gyrational or rotational symmetry. We say that a figure has rotational symmetry of order n if it coincides with itself exactly n times in one complete turn around its center. Thus, the figure coincides with itself after a rotation by an angle of $360°/n$.

Identify the order of rotational symmetry of the following figures:

⬠	✱	❀
⬡	✷	❉
⯃	✾	❂