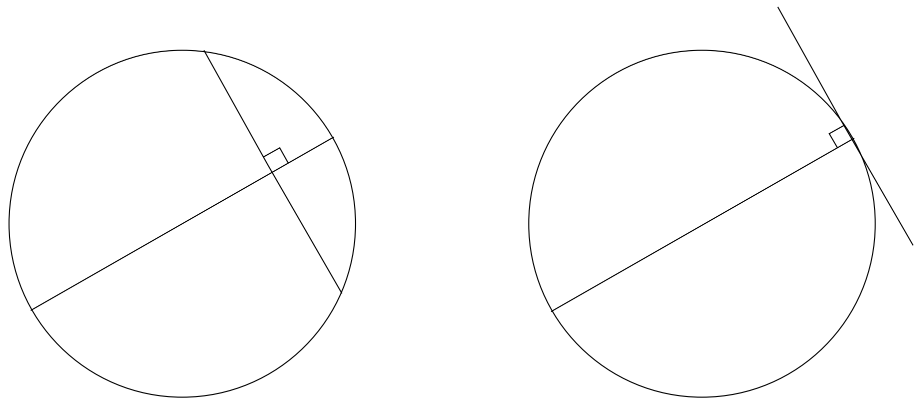
**Math 311. Inversion** A Candel
**Name:** CSUN Math

**¶ 1.** In many ruler and compass constructions it is important to know that the perpendicular bisector of a secant to a circle is a diameter of that circle. In particular, as a limiting case, this obtains that a a diameter to a circle is perpendicular to the lines tangent to that circle through the end points of the diameter.

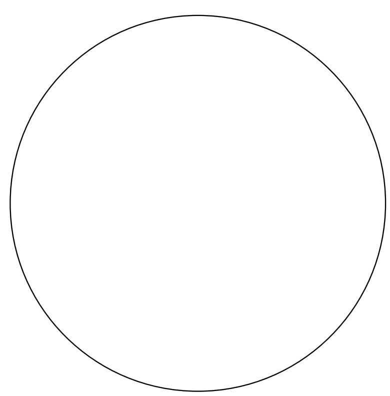


**¶ 2.** Find the center of a given circle. This can be accomplished as follows:

(a) Trace a line through two points $P$ and $Q$ on the circle

(b) Trace the perpendicular bisector to the line segment $\overline{PQ}$ to obtain a diameter $\overline{RS}$ of the given circle.

(c) The midpoint of the diameter $\overline{RS}$ is the center of the circle.



¶ **3.** Any three points that are not in the same line are in a unique circle. To find this circle, do the following.

(a) Let $A$, $B$, and $C$ be the given points.

(b) Trace the line segments $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$. Because the given points are not in a line, $ABC$ is a triangle. The circle to be found is the circumcircle of this triangle.

(c) Trace the perpendicular bisectors to each of the segments $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$.

(d) The bisector intersects in a single point $O$ which is the center of the circumcircle.

(e) The segments $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ all have the same length and serve as radius of the circumcircle.

¶ **4.** Given circles of $C$ and $C'$ of radius $r$ and $r'$ and exterior to each other, construct two lines tangent to them.

(a) Let $O$ and $O'$ be the centers of the given circles.

(b) Construct a point $P$ on the segment $\overline{OO'}$ dividing it into the ratio $r : r'$.

(c) Draw circles with diameter the segments $OP$ and $PP'$ (and centers on the segment $\overline{OO'}$). These circles intersect $C$ and $C'$ in points $R$, $S$, and $R'$, $S'$, respectively.

(d) Lines $\overleftrightarrow{PR}$ and $\overleftrightarrow{PS}$ are simultaneously tangent to the two circles $C$ and $C'$.

¶ **5.** Given two circle $C$ and $C'$ exterior to each other, find two tangent lines to them that do not intersect the segment joining the centers of the circles.

¶ **6.** Given two lines $L$ and $M$, and a radius $r$, find a circle $C$ with radius $r$ and tangent to the given lines $L$ and $M$.

¶ **7.** Inversion requires to find a tangent to a circle through a given point outside the circle. Given a point $P$ and a circle $C$ with center $O$, there are two such lines. To find them do the following.

(a) Find the midpoint $M$ of the segment $\overline{OP}$

(b) Construct the circle, $M^P$, with center $M$ through $P$.

(c) That circle $M^P$ meets the circle $C$ in points $A$ and $B$. The lines $\overleftrightarrow{AP}$ and $\overleftrightarrow{BP}$ are tangent to $C$ and go through $P$.

**¶ 8.** A transformation of the plane into itself is a rule that assigns to each point $P$ of the plane another point $P'$, called the image of $P$ under the transformation. A simple example of transformation is reflection of the plane on a given line $L$: a point $P$ on one side of $L$ has as its image the point $P'$ on the other side of $L$ such that the line $L$ is the perpendicular bisector of the segment $\overline{PP'}$. A transformation may leave some point fixed; in the case of a reflection this is true of every point on the line $L$.
Other examples of transformations of the plane are rotations, translations, similarities.

**¶ 9.** Another type of transformation of the plane is inversion on a circle. It was introduced in geometry by Jacob Steiner. It can be seen as certain kind of reflection on a circle, and thus sometimes are called circular reflections.
Let there be given a circle $C$ with center $O$ and radius $r$. The image of a point $P$ is the point $P'$ lying on the ray $\overrightarrow{OP}$, and such that the distances $OP$ and $OP'$ satisfy

$$OP \cdot OP' = r^2.$$

The points $P$ and $P'$ are said to be inverse points with respect to $C$. It follows that if $P'$ is the image of $P$, then $P$ is the image of $P'$. An inversion exchanges the inside of the circle with the outside of the circle: because $OP \cdot OP' = r^2$, we have that $OP < r$ if and only if $OP' > r$. It is also clear that the point lying on the circle $C$ remain fixed under inversion on $C$.
The rule above does not define the image of the center of the circle. It will make sense to think that the center $O$ has image the point $\infty$.

**¶ 10.** To find the image of a point $P$ under inversion on a circle $C$ with center $O$ we consider two cases: (1) the point $P$ is outside the circle $C$, and (2) the point $P$ is inside $C$.

(1) The point $P$ is outside $C$. Then

(1.a) Find two lines tangent to the circle $C$ and passing through $P$

(1.b) If $R$, $S$ are the points of tangency, the segment $\overline{RS}$ intersect $OP$ in the point $P'$.

(2) The point $P$ is inside $C$. Then

(2.a) Find the line $L$ perpendicular to $\overrightarrow{OP}$ through $P$.

(2.b) Line $L$ intersects $C$ in points $R$ and $S$.

(2.c) The tangent to $C$ through $R$ and $S$ intersect at $P'$.

¶ **11.** The most important property of an inversion is that it transforms lines and circles into lines and circle. More precisely, under inversion in a circle with center $O$:

(a) a line through $O$ becomes a line through $O$.

(b) a line not through $O$ becomes a circle through $O$.

(c) a circle through $O$ becomes a line not through $O$.

(d) a circle not through $O$ becomes a circle not through $O$.

Case (a) is obvious. Cases (b) and (c) are inverse of each other, so it suffices to verify one of them, say (b).

(a) Draw a circle $C$ with center $O$ and radius $r$.

(b) Draw a line $L$ not through $O$.

(c) Let $P$ be the intersection of $L$ with the perpendicular to $L$ through $O$.

(d) Let $P'$ be the inverse of $P$ on $C$, and let $C'$ be the circle with diameter equal to the segment $OP'$

(e) Let $A$ be any other point on the line $L$ and let $A'$ be its inverse on $C$.

(f) The right triangles $OA'P'$ and $OPA$ are similar because they have the same angle at $O$.

(g) Therefore, the proportion $\frac{OP}{OA} = \frac{OA'}{OP'}$ is true, and so $OA \cdot OA' = OP \cdot OP' = r^2$.

¶ **12.** The point $P'$ inverse to a given point $P$ with respect to a circle $C$ can be obtained with the use of the compass alone. Let the circle $C$ have center $O$ and radius $r$ and let $P$ be a point outside $C$.

(a) Construct the circle, $P^O$, with center $P$ and radius $OP$.

(b) Let $R$ and $S$ be the points where circle $C$ and circle $P^O$ intersect.

(c) With $R$ and $S$ as center, draw the circles with radius $r$.

(d) These two circles intersect in $O$ and in $P'$.

(e) Because the isosceles triangles $\triangle ORP$, $\triangle OP'R$ are similar, the proportions $OP'/OR = OR/OP$ hold.

¶ **13.** When the point $P$ is inside the circle $C$, the same construction of $P'$ works provided that the circle with center $P$ and radius $OP$ intersect the given circle $C$ in two points. This happens when $OP > r/2$. However, the following roundabout construction work in general.

(a) Find a whole number $n$ so that $nOP > r$, and then find a point $R$ on the ray $\overrightarrow{OP}$ so that $n \cdot OP = OR$.

(b) Let $R'$ be the inverse of $R$ on $C$.

(c) Let $P'$ be the point in $\overrightarrow{OP}$ such that $OP' = n \cdot OR'$.

(d) $P'$ is the inverse of $P$ on $C$. Indeed, $r^2 = OROR' = (n \cdot OP) \cdot (OP''/n) = OP \cdots OP'$.

¶ **14.** The method just outlined permits to find the center of a given circle with compass alone. Let $C$ be a circle and let $P$ be any point in $C$.

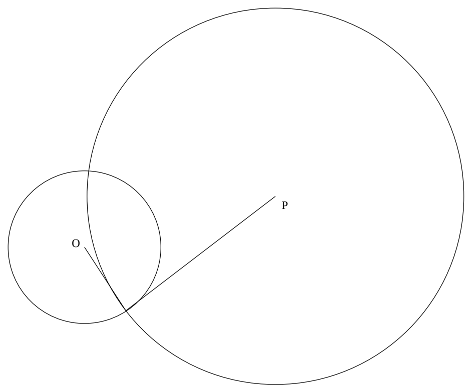
(a) Draw a circle $C_P$ with center $P$, and label $R$ and $S$ its point of intersection with $C$.

(b) With centers at $R$ and $S$, draw circles through $P$.

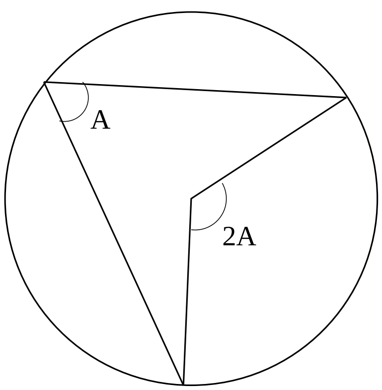
(c) Those two circles intersect at $P$ and at a new point $Q$.

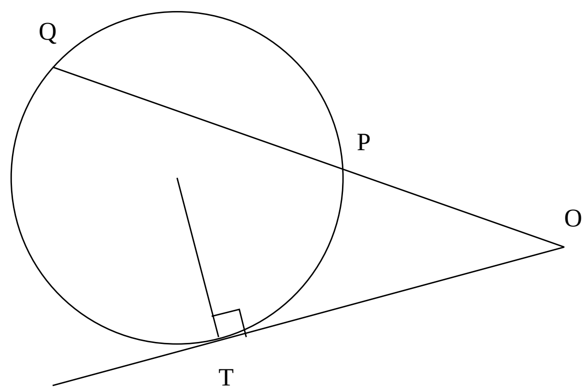
(d) The inverse of the point $Q$ with respect to the circle $C_P$ is the center of the original circle $C$.

**¶ 15.** The concept of "angle between lines" can be extended to "angle between curves at a given point of intersection." For example, if two circle intersect a a point $P$, the angle between those circles at $P$ is the angle between their tangent lines at that point $P$.
In particular, two circles are perpendicular at their point of intersection if the radius of one circle at that point is perpendicular to the tangent to the other circle at that point. This fact permits to easily construct a circle $C'$ with center $O'$ which is perpendicular to a given circle $C$ centered at $O$: Take the tangent to $C$ through $O'$ as the radius of the circle $C'$.




**¶ 16.** A fundamental fact of Euclidean geometry is the following (Euclid Proposition III-20): *In a circle, the angle at the center is twice the angle at the circumference, when the angles have the same circumference as base*.




This has many important consequences, one of which is relevant here: A tangent segment $\overline{OT}$ to a circle and a secant $\overline{OPQ}$ to the same circle satisfy the relation $OP \cdot OQ = OT^2$. This says that the points $P$ and $Q$ are inverse of each other with respect to the circle with center $O$ and radius $OT$.

¶ **17.** Even when inversions greatly distort figures, they do have an important property: they preserve angles. This means in particular that if two lines or a line and a circle are perpendicular or tangent at a given point, their images under an inversion will be similarly perpendicular or tangent at that point.
For example, consider inversion with center $O$, and consider the family of all circles that go through $O$ and through a given point $P$ in the plane. Their images under inversion with center $O$ will be lines that cross at the point $P'$. These lines are perpendicular to all the circles with center $P'$, and thus these circles correspond, under inversion to circles that bear little resemblance to them.