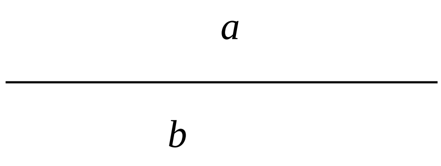
Math 311. Ruler and Compass Algebra A Candel

Name: CSUN Math

Using ruler and compass we will construct the sum difference, product and quotient of the lengths of two given segments. The methods are simple and visually appealing: they show how algebraic operations can be represented geometrically. They can be used to develop number sense and measurement intuition, as well as to explore the notions of congruence and similarity.
Incidentally, the fact that the sum, difference, product, and quotient of two numbers that can be constructed with ruler and compass is again a number that can be constructed with ruler and compass proves that the collection of all constructable numbers is a *field*.

¶ **1. Sum and difference** Given segments of lengths a and b, constructing segments of lengths $a + b$ and $a - b$ is not too difficult. For the sum, simply copy the segments on a straight line, one beginning where the other ends. For the difference, copy both segments beginning at the same point.
Use the following segments with lengths a and b to construct a segment of length $a + b$ and a segment of length $a - b$.



¶ **2. Product** Given segments of lengths a and b, constructing a segment of length ab is more challenging than constructing one of length $a + b$. While this latter construction was essentially a one dimensional affair, the former seems to require two dimensions.
Suppose that we are given segments of lengths a and b, and the unit segment, as drawn below. Do these steps, then answer the two question below:

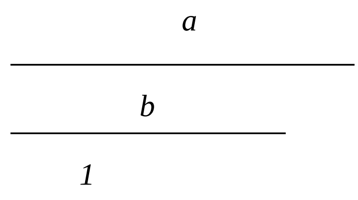
(a) Draw a ray O and placing on it the unit segment $\overline{OA}$

(b) On this ray, mark P so that $\overline{AP}$ has length a.

(c) On another ray from O mark a point B so that $\overline{OB}$ has length b.

(d) Draw a line through P parallel to $\overline{AB}$.

(e) Let Q denote the point of intersection of $\overrightarrow{OB}$ with that line in (d).



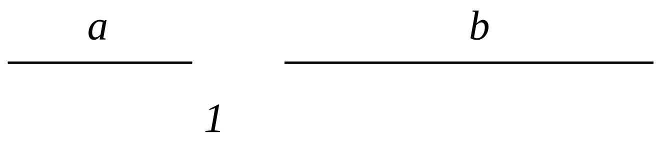
Let x denote the length of segment $\overline{BQ}$.

(i) Why is the proportion $\frac{b}{1} = \frac{x}{a}$ true?

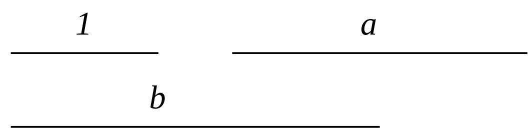
(ii) What does x equal?

¶ **3.** (a) Notice that in the example that you have just worked out, both length $a > 1$ and $b > 1$. How does the product ab compare to a and b?

(b) Suppose that the lengths $a < 1$ and $b < 1$, as in the figure below. How does ab compare to ab?



¶ **4. Quotient** Once you understand the method for constructing a segment of length the product of the lengths of two given segments, the methods for constructing a segment of length the quotient of the lengths of two given segments should be easy. It only requires you to be careful in identifying the placement of the numerator and denominator. Given the segments of lengths a and b below, write down the steps (as in Problem 2) to follow in order to construct a segment of length $\frac{a}{b}$.



(a) ____________________

(b) ____________________

(c) ____________________

(d) ____________________

(e) ____________________

(f) ____________________

(g) ____________________

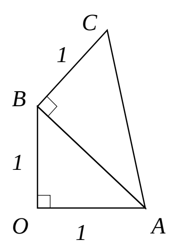
¶ **5. Algebraic expressions** We can combine the four algebraic operations to obtain geometric representations of general algebraic expressions.
Given the (unlabeled) segments below, assign to them lengths 1, a, b, c, and then construct a segment of length $\dfrac{ab+c}{a-c}$.

¶ **6. Construction of radical lengths** We have learned how to construct the sum, difference, product, and quotient of the lengths of two given segments using ruler and compass. It turns out that, given a segment of length a, it is also possible to construct a segment of length $\sqrt{a}$.
We start by constructing a segment of length $\sqrt{2}$ out of the unit segment, and then by iteration of the method we will construct radicals $\sqrt{n}$ of any whole number n.

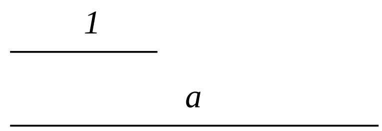
(a) Consider the triangle $\triangle OAB$ in the figure below. How long is its hypotenuse AB?

(b) In the right triangle $\triangle ABC$, how long is the hypotenuse $\overline{AC}$?

(c) (Pythagorean Spiral) Continue constructing right triangles by iterating the basic construction in this figure. At each step and over the last triangle constructed, construct a right triangle whose longer leg is the hypotenuse of the previous triangle, and whose shorter leg has length 1. Repeat until you have a right triangle with hypotenuse of length $\sqrt{7}$.



¶ **7.** It turns out that to construct a radical $\sqrt{a}$ it is not necessary to know the specific value of a, and in fact is not necessary to construct the "previous" value $\sqrt{a-1}$. Suppose that we are given, besides the unit segment, a segment of length a, as in the figure below.



Do the following:

(a) On a ray stating at O, mark off a point A so that the segment $\overline{OA}$ has length 1.

(b) On that ray. and from A, mark off a point B so that $\overline{AB}$ has length a (and thus $\overline{OB}$ has length $1 + a$).

(c) Construct the midpoint, M, of the segment $\overline{OB}$.

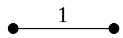
(d) Construct the circle with center M that goes through O.

(e) Construct a perpendicular to $\overline{OA}$ at A.

(f) Label C one of the points of intersection of that perpendicular in Step (e) with the circle of Step (d).

(g) Use the Pythagorean Theorem to find the length of the segment $\overline{AC}$.

The arithmetical operations that we have studied in this and the previous handout can now be combined.

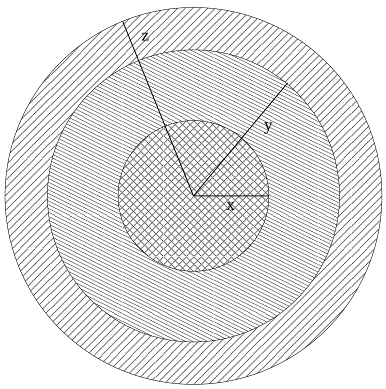
¶ **8.** This segment  has length 1. Use it to construct segments of the following lengths.

(a) $1 + \sqrt{3}$,

(b) $1 + 3\sqrt{2}$,

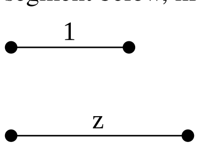
(c) $\sqrt{4 + 2\sqrt{2}}$.

¶ **9** (Trisecting a circular garden, [2]). We wish to divide a garden of circular shape into three concentric figures of equal area: one disk and two annuli, as in the figure below.



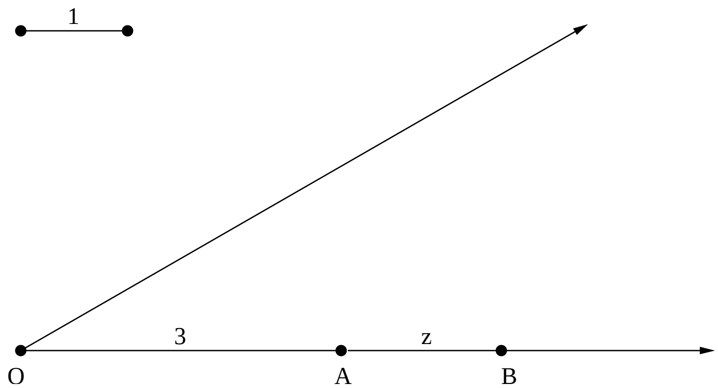
(a) The big circle has diameter z. The inner circle has diameter x. If the area of the small circle is one-third of the area of the big circle, what is the (algebraic) relationship between x and z? (Hint: the area of a circular disk or radius r is πr^2.)

(b) To construct the length x from the length z, we write the relationship in (a) in the form $\frac{x}{\sqrt{3}} = \frac{z}{3}$. Using the unit segment below, first construct a segment of length 3 and then one of length $\sqrt{3}$.



(c) In the figure below there are two rays from the point O: the lower ray with points A and B marked off, and the upper ray with no points marked off. On the lower ray, segment $\overline{OA}$ has length 3 and segment $\overline{AB}$ has length z. Find, on the upper line, mark off a point C so that the segment $\overline{OC}$ has length $\sqrt{3}$.

(d) Draw the segment $\overline{BC}$ and then construct the line parallel to $\overline{BC}$ through the point C and intersecting $\overrightarrow{OC}$ at D. What is the length of $\overline{CD}$? Why?



¶ **10.** The same method can be used to construct the radius y of the middle circle.

(a) How are the areas of the disks of radius z and y related?

(b) What algebraic relationship expresses that relation?

Bibliography

[1] C. Kinsey and T. Moore, *Symmetry, Shape, and Space*, Key College Publishing, 2002

[2] Alfred Posamentier, *Making Geometry come Alive*, Corwin Press, 2000.