The mathematical theory of ruller and compass constructions consists on performing geometric operation with a ruler and a compass. Any construction starts with two given points, or equivalently a segment (the unit), and uses the ruler and compass to construct new points. The ruler has no markings on it, and it is thus used for tracing a line passing two any pair of given points. The compass is used for tracing a circle with center a given point and passing through another given point.
All ruler and compass constructions thus start with two given points, and consist of repeated application of five basic constructions using the points, lines and circles that have already been constructed. These are:

(RC1) Constructing the line through two existing points.

(RC2) Constructing the circle through one point with center another point.

(RC3) Constructing the point which is the intersection of two existing, non-parallel lines.

(RC4) Constructing the one or two points in the intersection of a line and a circle (if they intersect).

(RC5) Constructing the one or two points in the intersection of two circles (if they intersect).

Notation We will use the following notation: for points A and B, let $\overleftrightarrow{AB}$ denote the line through A and B, let $\overrightarrow{AB}$ denote the ray with vertex A that passes through B, let $\overline{AB}$ denote the segment with endpoints A and B, and let AB denote the length of $\overline{AB}$. Furthermore, let A^B denote the circle through B with center A.
A ruler and compass construction will ususally consist of several steps like the ones above, which must be performed in a precise order. At each step new points appear as intersection points of previously constructed circles and lines. However, it is usually impossible to deduce the spets and order just by looking at the final result, so we will need to write down the steps one by one, so that anyone may reproduce them at a latter time. We can record those steps in plain English, or we may use the following schematic notation:

	p, q	
	A, B	

to denote the operation "let A, B be the points of intersection of the figures (circles or lines) p and q." If there are more restrictions on the figures p and q or on the points A and B, these are written into the scheme.

Constructions Many of the ruler and compass construction problems appear in Euclid's *Elements*. This is a work consisting of thirteen books, and constructions appear there as oppositions: a statement, which is followed by a solution and by a proof. We will explore some of them below.

¶ **1** (Euclid, Book I, Proposition 1). *Problem.* Given points A and B, construct a point C such that triangle $\triangle ABC$ is equilateral.

Construction. Draw the circles A^B and B^A. These circle intersect because the distance between their centers, AB, is smaller than the sum of their radii, $2AB$. If C is one of their intersection points, then $\triangle ABC$ is equilateral.

Proof. We have $AB = AC$ and $BA = BC$ because radii of the same circle are congruent. Therefore $AB = BC = CA$.

¶ **2.** *Problem.* Given two points A and B, construct the perpendicular bisector of the line segment AB.

Solution. Construct the circles A^B and B^A. These two circles meet at two points, E and F, on opposite sides of the line AB. Therefore, the line EF and the line AB intersect at a point C, which is the midpoint of AB.

Proof. Triangles $\triangle AEF \cong \triangle BEF$ because $AE = AF = AB$ and $BA = BE = BF$, and therefore $\angle AEC = \angle AEF = \angle BEF = \angle BEC$. Thus $\triangle AEC \cong \triangle BEC$, by SAS, and so $\angle ACE = \angle BCE$. Since they are complementary, each is $90°$.

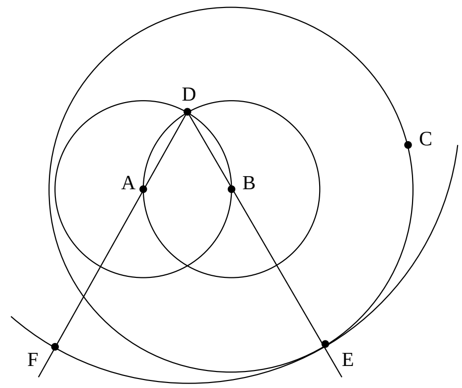
¶ **3.** [Euclid, Book I, Proposition 2]

Problem. Construct a segment congruent to a given segment and with a given point as one of its end points.

Solution. Let A be the given point and $\overline{BC}$ the given segment. We want to construct a point F such that $\overline{AF} \cong \overline{BC}$. The following scheme

A^B, B^A	$B^C, \overrightarrow{DB}$	$D^E, \overrightarrow{DA}$	F
D	E	F	

constructs such point F. The figure on the left illustrates this construction scheme. Next we prove that F has the advertised property.



Proof. We have $DF = DA + AF$ and $DE = DB + BE$. We also have $BE = BC$ and $DF = DE$ because radii of the same circle are equal. Therefore $DA + AF = DA + BC$ and so $AF = BC$.

¶ 4 (Euclid, Book I, Proposition 3). *Problem.* Given ray $\overrightarrow{AB}$ and segment $\overline{CD}$, construct a point H on $\overrightarrow{AB}$ such that $\overline{AH} \cong \overline{CD}$.

Hint. Use Problem 3 to construct a point G such that $\overline{AG} \cong \overline{CD}$. Let H be the intersection of A^G with $\overrightarrow{AB}$. The construction scheme is:

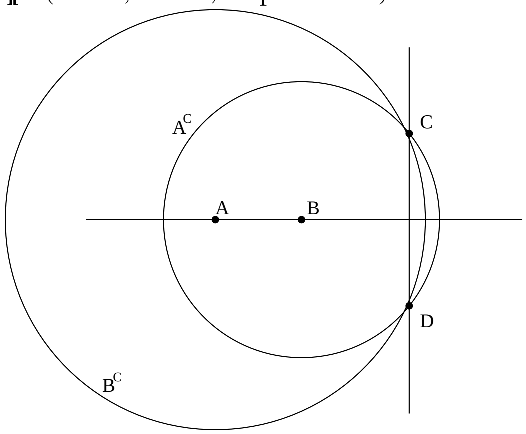
and the figure is:

¶ **5** (Euclid, Book I, Proposition 11). *Problem.* Given points A and B, construct the perpendicular to the segment $\overline{AB}$ at A.

Solution. In symbols:

$A^B, \overleftrightarrow{AB}$	B^C, C^B	$\overleftrightarrow{DA}$
C, B	D	

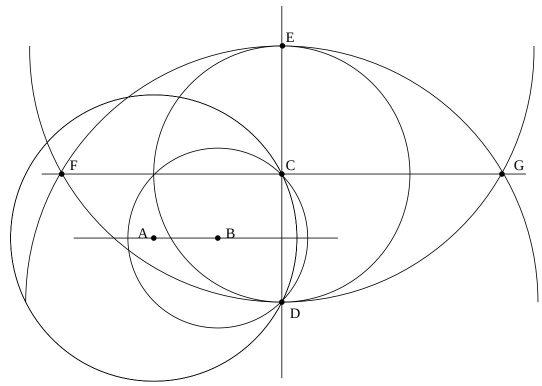
¶ **6** (Euclid, Book I, Proposition 12). *Problem.* Construct the perpendicular to a line through a point off that line.



¶ **7.** This construction plays a key role in ruler and compass arithmetic. Given a line and a point off it, construct a line through this point parallel to that line.

Solution. Let $\overleftrightarrow{AB}$ be the given line and C be a point off it. The construction scheme is

A^C, B^C	C^D, $\overleftrightarrow{CD}$	D^E, E^D	$\overleftrightarrow{FG}$
D	E	F, G	



¶ **8.** Given segments $\overline{AB}$ and $\overline{CD}$, construct a segment of length $AB + CD$.

Solution. Let E be the intersection of the line through D parallel to $\overleftrightarrow{BC}$ and the line through B parallel to $\overleftrightarrow{CD}$. Let F be the point of intersection of the circle B^E and the line $\overleftrightarrow{AB}$ that is not between A and B. Then $BF = BE = CD$ and $AF = AB + BF = AB + CD$.

¶ **9.** Given a segment $\overline{AB}$ and a whole number n, construct a segment of length AB/n.

Solution. Construct a point C off the line AB. On the ray $\overrightarrow{AC}$, construct points $n-1$ more points D, E, $\ldots$ such that $AC = CD = DE = \cdots$. If Z is the last point in the sequence $A, C, \cdots$, then construct the parallels to the line $\overleftrightarrow{ZB}$.

¶ **10.** You can now combine these two construction to obtain that given any fraction $\frac{p}{q}$, where p and q are whole numbers, it is always possible to construct two points A and B such that the length of the segment $\overline{AB}$ is precisely $\frac{p}{q}$.

(a) Given a segment of unit lenght, construct a segment of lenght $\frac{2}{3}$.

(b) The segment $\overline{AB}$ has length 3. Use it to construct a segment of length $\frac{1}{2}$.

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