Problem 1. Let f and g be integrable on $[a, b]$, and let c be a constant.

(a) Prove that cf is integrable on $[a, b]$ and $\displaystyle\int cf = c\int_a^b f$.

(b) Prove that $f + g$ is integrable on $[a, b]$ and $\displaystyle\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

Problem 2. If f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and $\displaystyle\left|\int_a^b f\right| \leq \int_a^b |f|$.

Problem 3. Evaluate without doing any computations

(a) $\displaystyle\int_{-1}^1 x^3\sqrt{1 - x^2}\, dx$.

(b) $\displaystyle\int_{-1}^1 (x^5 + 1)\sqrt{1 - x^2}\, dx$.

Problem 4. Suppose that f and g are integrable on $[a, b]$. If P is a partition of $[a, b]$, let M_i' and m_i' the appropriate sup's and inf's for f on the intervals of P, define M_i'' and m_i'' similarly for g, and define M_i and m_i similarly for the product fg.

Assume that $f(x) \geq 0$ and $g(x) \geq 0$ for all x in $[a, b]$.

(a) Prove that $M_i \leq M_i' M_i''$ and $m_i \geq m_i' m_i''$.

(b) Prove that

$$U(P, fg) - L(P, fg) \leq \sum_{i=1}^n (M_i' M_i'' - m_i' m_i'')(t_i - t_{i-1}).$$

(c) Use the fact that f and g are bounded (so that $|f(x)| \leq M$ and $|g(x)| \leq M$, for all x in $[a, b]$, to prove that

$$U(P, fg) - L(P, fg) \leq M\{U(P, f) + U(P, g) - L(P, f) - L(P, g)\}$$

(d) Prove that fg is integrable.

(e) Remove the condition that $f(x) \geq 0$ and $g(x) \geq 0$ on $[a, b]$.

Problem 5. (a) (Schwarz Inequality) Prove that

$$\left(\sum_{i=1}^n x_i y_i\right)^2 \leq \sum_{i=1}^n x_i^2 \cdot \sum_{i=1}^n y_i^2 \tag{$*$}$$

for real numbers $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$. There are many proofs available; one of them starts by first establishing the identity

$$\sum_{i=1}^n x_i^2 \cdot \sum_{i=1}^n y_i^2 = \left(\sum_{i=1}^n x_i y_i\right)^2 + \sum_{i<j} (x_i y_j - x_j y_i)^2.$$

(b) Prove that equality in $(*)$ holds if and only if there is a real number λ such that $x_i = \lambda y_i$ for all $i = 1, \cdots, n$.

(c) (Cauchy-Schwarz inequality) Suppose that f and g are integrable on $[a, b]$. Prove that

$$\left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right)\left(\int_a^b g^2\right). \tag{$**$}$$

(d) If equality holds in $(**)$, is it necessarily true that $f = \lambda g$ for some real number λ? What if f and g are continuous?

Problem 6. Prove that if $f(x) = x^3$, then $\displaystyle\int_0^b f = \frac{b^4}{4}$, by considering upper and lower sums for partitions of $[0, b]$ into n equal subintervals, using the formula $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for the sum of the cubes of the first n natural numbers.

Problem 7. Decide which of the following functions are integrable on $[0, 2]$, and calculate the integral sum if you can.

(a) $f(x) = \begin{cases} x + [x], & x \text{ rational} \\ 0, & x \text{ not rational.} \end{cases}$

(b) f is the function shown in Figure 1 (set $f(0) = 0$).

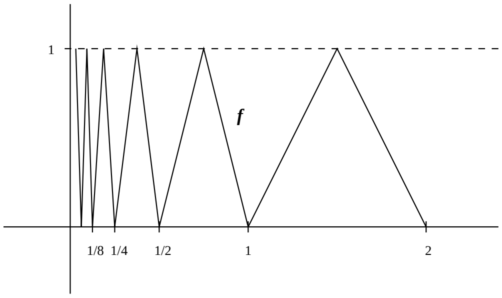

FIGURE 1

Problem 8. Find the areas of the regions bounded by

(a) The graphs of $f(x) = x^2$ and $g(x) = -x^2$ and the vertical lines through $(-1, 0)$ and $(1, 0)$.

(b)

Problem 9. (a) Prove that if f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then $\displaystyle\int_a^b f \geq 0$.

(b) Prove that if f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then $\displaystyle\int_a^b f \geq \int_a^b g$. (Warning: If you work hard on part (b), then you are waisting time.)

Problem 10. (a) Give an example of an f which is integrable on $[a, b]$, satisfies $f(x) \geq 0$ for all x, and $f(x) > 0$ for some x, and yet $\int_a^b f = 0$.

(b) Suppose that $f(x) \geq 0$ for all x in $[a, b]$ and f is continuous at x_0 in $[a, b]$ and $f(x_0) > 0$. Prove that $\int_a^b f > 0$. (Hint. It suffices to find a partition P for which the lower sum $L(f, P) > 0$.)

Problem 11. Prove that

$$\int_1^a \frac{1}{x}\,dx + \int_1^b \frac{1}{x}\,dx = \int_1^{ab} \frac{1}{x}\,dx.$$

Problem 12. Suppose that f is continuous on $[a, b]$ and that $\int_a^b fg = 0$ for all continuous functions g on $[a, b]$. Prove that $f = 0$.

Problem 13. Prove that

$$\int_1^a \frac{1}{x}\,dx + \int_1^b \frac{1}{x}\,dx = \int_1^{ab} \frac{1}{x}\,dx.$$

Problem 14. Prove that if f is continuous on $[a,b]$, then

$$\int_a^b f = (b-a)f(\xi)$$

for some number ξ in $[a,b]$; and show by example that continuity is essential.

The symbol $\lim\limits_{x\to\infty} f(x)$ means " the limit of $f(x)$ as x approaches ∞." We say that $\lim\limits_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there is a number M such that, for all x,

$$\text{if } x > M, \text{ then } |f(x) - L| < \varepsilon.$$

A similar definition applies to $\lim\limits_{x\to-\infty} f(x) = L$.

Problem 15. The limit $\lim\limits_{N\to\infty} \int_a^N f$, if it exists, is denoted by $\int_a^\infty f$ (or by $\int_a^\infty f(x)\cdot dx$), and called an "improper integral."

(i) Find $\int_1^\infty x^r \cdot dx$ if $r < -1$.

(ii) Prove that $\int_1^\infty \frac{1}{x} \cdot dx$ does not exits

(iii) Does $\int_0^\infty \frac{1}{1+x^2} \cdot dx$ exist?

The improper integral $\int_{-\infty}^a f$ is defined as $\lim\limits_{N\to-\infty} \int_N^a f$, as expected, but another kind of improper integral $\int_{-\infty}^\infty f$ is defined as $\int_0^\infty f + \int_{-\infty}^0 f$, provided both improper integrals exist.

(iv) Prove that $\int_{-\infty}^\infty \frac{1}{1+x^2} \cdot dx$ exist.

(v) Prove that $\lim\limits_{N\to\infty} \int_{-N}^N x \cdot dx$ exists, but the improper integral $\int_{-\infty}^\infty x \cdot dx$ does not exist.

(vi) (Not required) Prove that the improper integral $\int_\pi^\infty \frac{\sin x}{x} \cdot dx$ exists, but $\int_\pi^\infty \frac{|\sin x|}{x} \cdot dx$ does not exist.

Problem 16. There is another kind of improper integral in which the interval is bounded but the function is unbounded.

(i) If $a > 0$ and $-1 < r < 0$, find $\lim\limits_{\varepsilon\to 0+} \int_\varepsilon^a x^r \cdot dx$. This limit is denoted $\int_0^a x^r \cdot dx$, even though the function $f(x) = x^r$ is not bounded on $[0,a]$ (for $-1 < r < 0$), no matter how we define $f(0)$.

(ii) Suppose that f is continuous on $[0,1]$. Find

$$\lim_{x\to 0+} x \int_x^1 \frac{f(t)}{t} \cdot dt.$$

(iii) (Not required.) The integral $\int_0^\infty \frac{1}{x^2+\sqrt{x}} \cdot dx$ does not fall into any of the two kind of improper integrals previously described in these problems. Can you give it a meaning? (Break up the interval $(0,\infty)$ at 1.)