Problem 1. (a) Prove, working directly from the definition, that if $f(x) = 1/x$, then $f'(a) = -1/a^2$, for $a \neq 0$.

(b) Prove that the tangent line to the graph of f at $(a, 1/a)$ does not intersect the graph of f, except at $(a, 1/a)$.

Solution. (a) By definition,

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x\to a} \frac{1/x - 1/a}{x - a} \\ &= \lim_{x\to a} \frac{a - x}{ax(x - a)} \\ &= \lim_{x\to a} \frac{-1}{ax} \\ &= \frac{-1}{a^2} \end{aligned}$$

(b) The tangent line to the graph of f at $(a, 1/a)$ has equation $y = (-1/a^2)x + (2/b)$. If this line intersects the graph of f at a point $P = (x, y)$, then the coordinates of P must satisfy

$$y = \frac{1}{x},$$

and

$$y = \frac{-x}{a^2} + \frac{2}{b}.$$

By equating these two identities and simplifying you obtain that $x = a$ and $y = 1/a$.

□

Problem 2. Discuss the differentiablility of $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} x^3 \sin \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Problem 3. Find f' if $f(x) = [x]$. (Here $[x]$ is the largest integer less than or equal to x.)

Solution. If a is an integer, then the function f is not differentiable at a.

If n is an integer, then on the interval $(n, n+1)$, the function f is constant with value $f(x) = n$ for all x in $(n, n+1)$. Hence $f'(x)$ for x in $(n, n+1)$.

Therefore, $f'(x)$ is defined if x is not an integer, and $f'(x) = 0$ for all those x. □

Problem 4. Let $f(a) = x^2$ if x is rational, and $f(x) = 0$ if x is not rational. Prove that f is differentiable at 0.

Problem 5. Let f be a function such that $|f(x)| \leq x^2$ for all x. Prove that f is differentiable at 0.

Solution. Note first that $|f(0)| \leq 0$, so in fact $f(0) = 0$. Moreover, because $|f(x)| \leq x^2$, if $x \neq 0$,

$$0 \leq \frac{|f(x)|}{|x|} \leq |x|,$$

hence

$$\lim_{x\to 0} \frac{|f(x) - f(0)|}{|x - 0|} = 0.$$

Now

$$-\frac{|f(x)|}{|x|} \leq \frac{f(x)}{x} \leq \frac{|f(x)|}{|x|},$$

so also

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = 0.$$

□

Problem 6. Prove that if f is even, then $f'(x) = -f'(-x)$. (A function f is even if $f(x) = f(-x)$.)

Solution. There are two ways of writing up the solution. A straightforward way:

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} \\ &= \lim_{h\to 0} \frac{f(x-h) - f(x)}{h} \qquad \text{(because } f \text{ is even)} \\ &= -\lim_{h\to 0} \frac{f(x-h) - f(x)}{-h} \qquad \text{(multiply and divide by } -1.) \\ &= -f'(-x) \end{aligned}$$

Another way: If J denotes the function $J(x) = -x$, then the fact that f is even may be expressed by saying that $(f \circ J)(x) = f(x)$. Taking derivatives on both sides, and using the Chain Rule on the left side,

$$f'(J(x)) \cdot J'(x) = f'(x).$$

Since $J'(x) = -1$, the proposed identity $-f'(-x) = f'(x)$ results. □

Problem 7. Find $f'(x)$ if

$$f(x) = \frac{1}{x - \dfrac{2}{x + \sin x}}.$$

Solution. □

Problem 8. Find $f'(x)$ for $-1 < x < 1$, if $f(x) = \sqrt{1 - x^2}$.

Problem 9. Suppose that a and b are two consecutive roots of a polynomial function f, but that a and b are not double roots. so that we can write $f(x) = (x-a)(x-b)g(x)$ where $g(a) \neq 0$ and $g(b) \neq 0$.

(a) Prove that $g(a)$ and $g(b)$ have the same sign.

(b) Prove that there is some number x with $a < x < b$ and $f'(x) = 0$.

(c) Now prove the same fact, even if a and b are multiple roots.

Solution. (a) If $g(a)$ and $g(b)$ have opposite sign, then $g(x_0) = 0$ for some x_0 in (a, b). Then, for this x, $f(x) = (x_0 - a)(x_0 - b)g(x_0) = 0$, contradicting that a and b are consecutive roots of f.
(b) Differentiate the expression $f(x) = (x-a)(x-b)g(x)$ to get

$$f'(x) = (x-b)g(x) + (x-a)g(x) + (x-a)(x-b)g'(x)$$

Then

$$f'(a) = (a-b)g(b) \qquad \text{and} \qquad f'(b) = (b-a)g(a).$$

By part (a), $g(a)$ and $g(b)$ have the same sign. Thus $f'(a)$ and $f'(b)$ have opposite sign; the mean value theorem implies that $f'(x) = 0$ for some x in (a, b). □

Problem 10. If f is differentiable three times and $f'(x) \neq 0$, the Schwarzian derivative of f at x is defined to be

$$\mathcal{S}f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

(a) Show that

$$\mathcal{S}(f \circ g) = [\mathcal{S}f \circ g] \cdot (g')^2 + \mathcal{S}g.$$

(b) Show that if $f(x) = \dfrac{ax+b}{cx+d}$, with $ad - bc \neq 0$, then $\mathcal{S}f = 0$. Consequently, $\mathcal{S}(f \circ g) = \mathcal{S}g$.

Solution. Write the chain rule as $(f \circ g)' = (f' \circ g) \cdot g'$, and use the rules for derivatives to obtain

$$\begin{aligned}(f \circ g)'' &= (f' \circ g)' \cdot g' + (f' \circ g) \cdot g'' \\ &= ((f'' \circ g) \cdot g') \cdot g' + (f' \circ g) \cdot g'' \\ &= (f'' \circ g) \cdot (g')^2 + (f' \circ g) \cdot g''\end{aligned}$$

and

$$\begin{aligned}(f \circ g)''' &= ((f'' \circ g) \cdot (g')^2 + (f' \circ g) \cdot g'')' \\ &= (f''' \circ g) \cdot (g')^3 + 3(f'' \circ g) \cdot g'g'' + (f' \circ g) \cdot g'''\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{D}(f \circ g) &= \frac{(f \circ g)'''}{(f \circ g)'} - \frac{3}{2}\Big(\frac{(f \circ g)''}{(f \circ g)'}\Big)^2 \\ &= \frac{(f''' \circ g)(g')^2}{f' \circ g} + 3\frac{(f'' \circ g)g''}{f' \circ g} + \frac{g'''}{g'} - \frac{3}{2}\Big(\frac{(f'' \circ g)g'}{f' \circ g} + \frac{g''}{g'}\Big)^2 \\ &= \Big[\frac{f'''}{g'} \circ g - \frac{3}{2}\Big(\frac{f'' \circ g}{f' \circ g}\Big)^2\Big] \cdot (g')^2 + \frac{g'''}{g'} - \frac{3}{2}\Big(\frac{g''}{g'}^2\Big) \\ &= [\mathcal{D}f \circ g] \cdot (g')^2 + \mathcal{D}g.\end{aligned}$$

(b) For $f(x) = \dfrac{ax+b}{cx+d}$ (with $ad - bc \neq 0$) we compute

$$f'(x) = \frac{a(cx+d) - c(ax+d)}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2},$$

$$f''(x) = \frac{-2c(ad-bc)}{(cx+d)^3},$$

and

$$f'''(x) = \frac{6c^2(ad-bc)}{(cx+d)^3}.$$

Therefore

$$\begin{aligned}\frac{f'''(x)}{f'(x)} - \frac{3}{2}\Big(\frac{f''(x)}{f'(x)}\Big)^2 &= \frac{6c^2}{(cx+d)^2} - \frac{3}{2}\Big(\frac{-2}{cx+d}\Big)^2 \\ &= 0.\end{aligned}$$

□

Problem 11. Let $a > 0$. Show that the maximum value of the function

$$f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x-a|}$$

is $(2+a)/(1+a)$.

Solution. The function f is differentiable on $(-\infty, 0)$, $(0, a)$ and (a, ∞). We can write

$$f(x) = \begin{cases} \dfrac{1}{1-x} + \dfrac{1}{1+a-x}, & x < 0 \\[2ex] \dfrac{1}{1+x} + \dfrac{1}{1+a-x}, & 0 < x < a \\[2ex] \dfrac{1}{1+x} + \dfrac{1}{1+x-a}, & a < x, \end{cases}$$

so

$$f'(x) = \begin{cases} \dfrac{1}{(1-x)^2} + \dfrac{1}{(1+a-x)^2}, & x < 0 \\[2ex] -\dfrac{1}{(1+x)^2} + \dfrac{1}{(1+a-x)^2}, & 0 < x < a \\[2ex] -\dfrac{1}{(1+x)^2} - \dfrac{1}{(1+x-a)}, & a < x, \end{cases}$$

Thus f is increasing on $(-\infty, 0]$ and decreasing on $[a, \infty)$, so the maximum of f on $\mathbf{R}$ is the maximum of f on $[0, a]$.

If $f'(x) = 0$ for x in $(0, a)$, then

$$-\frac{1}{(1+x)^2} + \frac{1}{(1+a-x)^2} = 0,$$

or

$$(1+x)^2 - (1+x-a)^2 = 0.$$

The only solution to this equation in $(0, a)$ is $x = a/2$.

Now

$$f(0) = f(a) = \frac{2+a}{1+a} > \frac{4}{2+a} = f(a/2),$$

so the maximum value is $(2+a)/(1+a)$. □

Problem 12. A function f is Lipschitz of order α at x is there is a constant C such that

$$|f(x) - f(y)| \le C|x-y|^\alpha \qquad (*)$$

for all y in an interval around x. The function f is Lipschitz of order α on an interval if $(*)$ holds for all x and y in the interval.

(a) If f is Lipschitz of order $\alpha > 0$ at x, then f is continuous at x.

(b) If f is differentiable at x, then f is Lipschitz of order 1 at x.

(c) If f is Lipschitz of order $\alpha > 1$ at x, then f is differentiable at x and $f'(x) = 0$.

Solution. (a) Given $\varepsilon > 0$, let

$$\delta = \sqrt[\alpha]{\frac{\varepsilon}{C}}.$$

If $|x-y| < \delta$, then

$$|f(x) - f(y)| \le C|x-y| < C\delta = \varepsilon.$$

(b) If f is differentiable at x, then

$$\lim y \to x \frac{f(y) - f(x)}{y - x} = f'(x)$$

Thus

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < 1$$

for all y in an interval around x. Hence

$$\left| \frac{f(y) - f(x)}{y - x} \right| < 1 + |f'(x)|$$

$$|f(y) - f(x)| \le (1 + |f'(x)|)|y - x|,$$

so we can choose $C = 1 + |f'(x)|$.

(c) Say $\alpha = 1 + \beta$ for some $\beta > 0$. Then the Lipschitz condition

$$|f(y) - f(x)| \le C|y - x|^\alpha = C|y - x||y - x|^\beta$$

implies

$$0 \le \left|\frac{f(y) - f(x)}{y - x}\right| \le C|y - x|^\beta$$

Taking limits as $y \to x$ we obtain

$$\lim_{y\to x}\left|\frac{f(y) - f(x)}{y - x}\right| = 0,$$

since $\lim_{y\to x}|y - x|^\beta = 0$ because $\beta > 0$. This implies that also

$$\lim_{y\to x}\frac{f(y) - f(x)}{y - x} = 0,$$

so that $f'(x) = 0$ □

Problem 13. [Cauchy's Mean Value Theorem] Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Then there is a number x in (a, b) such that

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x).$$

(Cf. Problem 8, pg 108 of textbook for hints.)

Solution. Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Then there is a number x in (a, b) such that

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x).$$

Proof. Let $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$. Then h is continuous on $[a, b]$ and differentiable on (a, b). Also $h(a) = f(a)g(b) - g(a)f(b) = h(b)$. Now apply Rolle's Theorem. □

Problem 14 (L'Hôpital's Rule)**.** Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and that $\lim_{x\to a} f'(x)/g'(x)$ exists. Then $\lim_{x\to a} f(x)/g(x)$ also exists and

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Solution. Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and that $\lim_{x\to a} f'(x)/g'(x)$ exists. Then $\lim_{x\to a} f(x)/g(x)$ also exists and

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Proof. The hypothesis that $\lim_{x\to a} f'(x)/g'(x)$ implies

(1) there is an interval $I = (a - \delta, a + \delta)$ such that $f'(x)$ and $g'(x)$ exists for all x in I, expect perhaps for $x = a$.

(2) in this interval, $g'(x) \ne 0$, with the possible exception of $x = a$.

Define $f(a) = g(a) = 0$, so that f and g are continuous at a. If $a < x < a + \delta$, then both the Mean Value Theorem and Cauchy's Mean Value Theorem apply to f and g on $[a, x]$. By the Mean Value theorem, we see that $g(x) \ne 0$, for if $g(x) = 0$, then $g'(y) = 0$ for some y in (a, x). By Cauchy's Mean Value Theorem, there is b_x in (a, x) such that

$$(f(x) - 0)g'(b_x) = (g(x) - 0)f'(b_x)$$

Now b_x approaches 0 as x approaches a. Since $\lim_{y\to a} f'(x)/g'(x)$ exists, it follows that

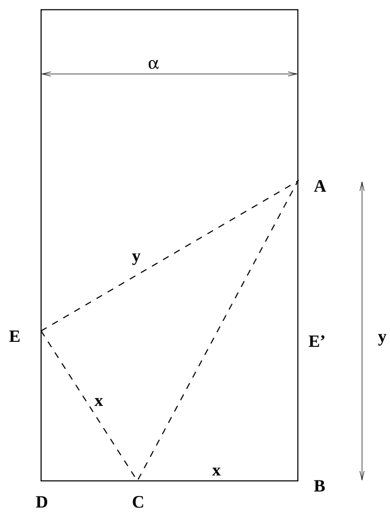
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(b_x)}{g'(b_x)} = \lim_{y\to a} f'(y)g'(y).$$

□

Problem 15. What is the largest area of any equilateral triangle enclosed in a unit square?

Problem 16. The lower right-hand corner of a page is folded over so that it just touches the left edge of the paper, as in the figure below If the width of the paper is very long, show that the minimum length of the crease is $3\sqrt{3}\alpha/4$.

Solution. If $x = \overline{BC}$ and $y = \overline{AB}$



then we have

$$\overline{ED} = \sqrt{x^2 + (\alpha - x)^2} = \sqrt{2\alpha x - \alpha^2} \qquad \text{from } \triangle EDC$$
$$\alpha^2 + (y - \overline{ED})^2 = y \qquad \text{from } \triangle EE'A,$$

so

$$\begin{aligned} \alpha^2 + (y - \sqrt{2\alpha x - \alpha^2})^2 &= y^2 \\ -y\sqrt{2\alpha x - \alpha^2} + \alpha x &= 0 \\ y^2(2\alpha x - \alpha^2) &= \alpha^2 x^2 \\ y^2 &= \frac{\alpha^2 x^2}{2\alpha x - \alpha^2} = \frac{\alpha x^2}{2x - \alpha}. \end{aligned}$$

The square of the length of the crease is

$$x^2 + y^2 = x^2 + \frac{\alpha^2 x^2}{2\alpha x - \alpha^2} = \frac{\alpha x^2}{2x - \alpha},$$

so the length is smallest when

$$0 = 6x^2(2x - \alpha) - 4x^3 = 8x^3 - 6x^2\alpha = x^2(8x - \alpha),$$

or $x = 3\alpha/4$. For this x the length is $3\sqrt{3}\alpha/4$. □