THE RECTILINEAR CROSSING NUMBER OF K_n: CLOSING IN (OR ARE WE?)

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Abstract. The calculation of the rectilinear crossing number of complete graphs is an important open problem in combinatorial geometry, with important and fruitful connections to other classical problems. Our aim in this work is to survey the body of knowledge around this parameter.

1. Introduction

In a *rectilinear* (or *geometric*) drawing of a graph G, the vertices of G are represented by points, and an edge joining two vertices is represented by the straight segment joining the corresponding two points. Edges are allowed to cross, but an edge cannot contain a vertex other than its endpoints. The *rectilinear* (or *geometric*) crossing number $\overline{\text{cr}}(G)$ of a graph G is the minimum number of pairwise crossings of edges in a rectilinear drawing of G in the plane.

1.1. The relevance of $\overline{\text{cr}}(K_n)$.

As with every graph theory parameter, there is a natural interest in calculating the rectilinear crossing number of certain families of graphs, such as the complete bipartite graphs $K_{m,n}$ and the complete graphs K_n. The estimation of $\overline{\text{cr}}(K_n)$ is of particular interest, since $\overline{\text{cr}}(K_n)$ equals the minimum number $\square(n)$ of convex quadrilaterals defined by n points in the plane in general position; the problem of determining $\square(n)$ belongs to a collection of classical combinatorial geometry problems, the so-called Erdős-Szekeres problems.

Another important motivation to study $\overline{\text{cr}}(K_n)$ is its close connection with the celebrated Sylvester Four Point Problem from geometric probability. Sylvester asked what is the probability that four points chosen at random in the plane form a convex quadrilateral [30]. After it became clear that this is an ill-posed question [31], Sylvester put forward a related conjecture. Let R be a bounded convex open set in the plane with finite area, and let $q(R)$ be the probability that four points chosen randomly from R define a convex quadrilateral. Then (Sylvester's Conjecture [21]) $q(R)$ is minimized when R is a circle or an ellipse, and maximized when R is a triangle. This conjecture was proved by Blashke in 1917 [17]. Scheinerman and Wilf addressed in [28] the general problem when R is not required to be convex. It is easy to see that in this case $q(R)$ can be made arbitrarily close to 1 by choosing R to be very thin annulus. The remaining problem is to determine the infimum $q_* := \inf q(R)$, taken over all open sets R with finite area. Scheinerman and Wilf

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established the striking connection

$$q_* = \lim_{n\to\infty} \frac{\overline{\mathrm{cr}}(K_n)}{\binom{n}{4}}, \tag{1}$$

thus inextricably linking the estimation of *Sylvester's Four Point Constant* q_* to the (asymptotic) behaviour of $\overline{\mathrm{cr}}(K_n)$.

As we shall see below, recent developments have unveiled a close relationship between $\overline{\mathrm{cr}}(K_n)$ and yet another classical combinatorial geometry parameter: the number of $(\leq k)$-edges in an n-point set.

1.2. **Purpose and timeliness of this survey.** Up until 2000, very little was known about $\overline{\mathrm{cr}}(K_n)$. Since then, our knowledge of this problem has seen a tremendous growth. Surprising and useful connections to other classical problems have been unveiled. The current estimates for $\overline{\mathrm{cr}}(K_n)$ have reached a point that would have seemed unlikely (to say the least) at the beginning of the previous decade.

For instance, before 2000 the ratio between the best lower and upper bounds for q_* was about 0.755; at the time of writing this survey, this ratio has been raised above 0.998. The implied success in our understanding of the problem cannot be understated —hence the "closing in" words in the title of this survey. Moreover, as we have already mentioned above and shall see below in more detail, the problem of estimating $\overline{\mathrm{cr}}(K_n)$ has turned out to be intimately related to other classical combinatorial geometry problems. Nowadays, anyone seriously interested in $(\leq k)$-edges or in halving lines, has no alternative but to take a careful look at the literature on $\overline{\mathrm{cr}}(K_n)$ that has been produced in the last seven or eight years.

On the more cautious side, we must also note that the steady progress achieved on the estimation of $\overline{\mathrm{cr}}(K_n)$, both from the lower and the upper bounds fronts, seems to have reached an impasse. To a researcher not too familiar with the field, the ratio 0.998 mentioned in the previous paragraph might signal an imminent closure on the determination of q_*. This is by no means the prevalent feeling among most (if not all) researchers actively working on this problem. Hardly any relevant new insights have been reported for some time. This humbling reality prompted us to include a word of caution ("or are we?") in the title of this survey.

With this in mind, it makes sense to sit down and reflect on what has been done, to highlight the key developments, and to record the state-of-the-art of the problem. We also see this as an opportunity to candidly (and, at times, informally) explain the obstacles that seem to prevent any further substantial progress with the current techniques, in the hopes that this will foster the development of refined or substantially novel techniques to attack this fundamental problem.

1.3. **Structure of this survey.** The problem of estimating $\overline{\mathrm{cr}}(K_n)$ breaks into the two problems of establishing upper and lower bounds for this parameter, with the problem of finding exact values of $\overline{\mathrm{cr}}(K_n)$ lying, evidently, within both realms.

Before moving on to separate discussions on the problems of lower- and upper-bounding $\overline{\mathrm{cr}}(K_n)$, we shall review one of the main foundations behind our current knowledge of $\overline{\mathrm{cr}}(K_n)$. The Rectilinear Crossing Number project (RCN), led by Aichholzer, has been a fruitful source of inspiration as well as an invaluable tool for establishing results and testing conjectures. In Section 2 we describe the nature and reach of the RCN project which, as we will see, has both a claim and an impact on both the lower- and the upper-bounding fronts.

In Section 3 we give an overview of the state-of-the-art of the problem of lower bounding $\overline{\text{cr}}(K_n)$ circa 2003.

Besides Aichholzer's RCN project, there seems to be a general consensus on the other main foundation behind our current knowledge of $\overline{\text{cr}}(K_n)$. A major breakthrough was achieved around 2003, when two independent teams of researchers elucidated the close connection between $\overline{\text{cr}}(K_n)$ and the number of $(\leq k)$-edges in an n-point set [4, 26]. A good estimate on the number of such $(\leq k)$-edges, also given in these papers, yielded an impressively improved lower bound on $\overline{\text{cr}}(K_n)$. We devote Section 4 to a review of these cornerstone results.

In Section 5 we overview the subsequent efforts to refine the bounds for the number of $(\leq k)$-edges given in [4] and [26], in the quest for improved lower bounds for $\overline{\text{cr}}(K_n)$.

In Section 6 we discuss the different approaches to establishing upper bounds for $\overline{\text{cr}}(K_n)$.

Section 7 contains a brief summary of the state-of-the-art of the problem at the time of writing this survey. We present the current best estimates (lower and upper bounds) for q_*, as well as an annotated table with the values of n for which the exact value of $\overline{\text{cr}}(K_n)$ is known.

We conclude this survey by reflecting on some possible future developments around this fundamental problem. We discuss the difficulties that lie behind our current impasse, and outline a somewhat promising approach that may pave the way towards future improvements.

2. The Rectilinear Crossing Number project

Around 2000, a team of researchers led by Aichholzer undertook the task of building databases with all the distinct (up to order type equivalence; see below) n-point configurations in general position, for $n \leq 10$ [10, 14, 25]. The raw knowledge of all possible n-point configurations put Aichholzer and his collaborators in a position to explore in depth several classical combinatorial geometry problems. In particular, it allowed for the exact calculation of $\overline{\text{cr}}(K_n)$ for small values of n.

The criterion used by Aichholzer et al. to discriminate if two collections of points are non-isomorphic is based on the concept of order types. Consider an (ordered) n-point set $P = \{p_1, p_2, \ldots, p_n\}$ in general position. To each three integers i, j, k with $1 \leq i < j < k \leq n$, associate a *sign* (or *order type*) $\text{sign}(ijk)$ according to the following rule. If as we traverse the triangle defined by p_i, p_j, and p_k by following the edges $\overline{p_i p_j}$, $\overline{p_j p_k}$, and $\overline{p_k p_i}$ in the given order, the resulting closed curve has a clockwise orientation, then let $\text{sign}(ijk) := +$. Otherwise, let $\text{sign}(ijk) := -$. The collection of the order types of all the triples of points of P is the *order type* of P. Now let Q be another n-point set in general position. If the elements of Q can be ordered $\{q_1, q_2, \ldots, q_n\}$ so that the order types of P and Q are the same, then P and Q are *order type equivalent* (under the mapping $p_i \mapsto q_i$ for $i = 1, 2, \ldots, n$). We simply say that P and Q *have the same order type.*

Order type equivalence is a natural isomorphism criterion for point sets in general position. For crossing number purposes, it is certainly the relevant paradigm. Indeed, suppose that P and Q have the same order type. Then there is a bijection from the points of P to the points of Q so that four points in P span a convex quadrilateral if and only if the corresponding four points in Q span a convex quadrilateral. Conversely, if this last condition holds, then P and Q have the same order type.

Aichholzer et al. constructed the complete database of all distinct order types on n points, for all $n \leq 10$. As an application, they verified that $\overline{\text{cr}}(K_{10}) = 62$ (this had been proved by Brodsky et al. in [18]).

Without building the complete database for $n = 11$, the information gathered by Aichholzer et al. for $n \leq 10$ allowed them to calculate $\overline{\text{cr}}(K_{11})$ and $\overline{\text{cr}}(K_{12})$. To achieve this, taking their database for 10 points as a starting point, they analyzed (for $m = 10$, and then for $m = 11$) which m-point order types may possibly be extended to $(m+1)$-point sets that correspond to crossing-minimal drawings of K_{m+1}.

The determination of the rectilinear crossing numbers of K_{11} and K_{12} marks the beginning of the Rectilinear Crossing Number project (RCN). As one of the major achievements of the RCN, Aichholzer developed some impressively accurate heuristics to generate geometric drawings of K_n with few crossings. Aichholzer set up a web page [8] to keep track of the best geometric drawings of K_n available, as well as of the number of distinct (up to order type equivalence) drawings achieving the current minimum.

The results reported by Aichholzer in [8] have had a major lasting impact in the field. As new results and techniques to find improved lower bounds have become available (see Sections 4 and 5), it has been possible to determine the exact value of $\overline{\text{cr}}(K_n)$ for more values of n (see Section 7). The outstanding quality of the upper bounds obtained by Aichholzer is evidenced by the fact that the drawings reported in [8] turned out to be crossing optimal for all $n \leq 27$ and for $n = 30$ (for $n = 28$ and 29 the exact value of $\overline{\text{cr}}(K_n)$ is still unknown). At the time of writing this survey, the best upper bounds available (see Section 6) are obtained from constructions that build upon "base" drawings of K_n for relatively small values of n. As a further evidence of the influence of the RCN, we note that the base drawings used have been obtained by small modifications of drawings given in [8].

As a final note, let us mention that Aichholzer and Krasser subsequently completed the database of all distinct order types of 11-point sets [15, 9]. Using this database as a startpoint, they were able to compute $\overline{\text{cr}}(K_n)$ for all $n \leq 17$. Building the complete database of all the order type nonequivalent 12-point sets seems to be an unfeasible task; not only it is estimated that the storage of these 12-point sets would require several petabytes of memory, but there are also some important technical difficulties.[1]

3. Lower bounds I: before 2004

In a paper published in 1972, Guy [23] gave the exact value of $\overline{\text{cr}}(K_n)$ for $n \leq 9$. Almost thirty years later, Brodsky, Durocher, and Gethner [18] pushed the existing techniques to their limit, and introduced some clever new arguments, to calculate the exact value of $\overline{\text{cr}}(K_{10})$.

As one of the first results of the Rectilinear Crossing Number project (see Section 2), Aichholzer, Aurenhammer, and Krasser [11] gave computer-assisted proofs that $\overline{\text{cr}}(K_{11}) = 102$ and $\overline{\text{cr}}(K_{12}) = 153$.

[1] Aichholzer, personal communication.

Because each of the n subsets of size $n-1$ of an n-point set P has at most $\overline{\text{cr}}(K_{n-1})$ crossings, and each crossing of P appears in exactly $n-4$ such subsets, it follows that $(n-4)\overline{\text{cr}}(K_n) \geq n\overline{\text{cr}}(K_{n-1})$. This is equivalent to

$$1 \geq \frac{\overline{\text{cr}}(K_n)}{\binom{n}{4}} \geq \frac{\overline{\text{cr}}(K_{n-1})}{\binom{n-1}{4}},$$

which shows that the Sylvester's Four Point Constant q_* defined in (1) actually exists. Starting from a lower bound for $\overline{\text{cr}}(K_m)$ for any fixed m, one can obtain a lower bound for $\overline{\text{cr}}(K_n)$ for every $n > m$ (and consequently a lower bound for q_*) by iterating $\overline{\text{cr}}(K_n) \geq \lceil \overline{\text{cr}}(K_{n-1})n/(n-4)\rceil$. This technique was used by Brodsky et al. [18] with $\overline{\text{cr}}(K_{10}) = 62$ to show that $q_* > 0.3001$. Adding to this argument the fact that $\overline{\text{cr}}(K_n)$ and $\binom{n}{4}$ have the same parity when n is odd (this easily follows from (2) but was proved for any non-necessarily rectilinear drawing of K_n by Eggleton and Guy [22]), and using $\overline{\text{cr}}(K_{11}) = 102$, Aichholzer et al. [11] showed that $q_* > 0.3115$.

Building upon ideas from Welzl [35] and Wagner and Welzl [33], Wagner [32] used a completely novel approach to show that $q_* > 0.3288$. Wagner's work is particularly significant, since it deviates from the traditional approach of lower bounding q_* by using a particular lower bound and a counting argument. Indeed, the ideas in [32] are prescient of the revolutionary approach that will be reviewed in the next section.

4. Lower bounds II: the breakthrough

Our understanding of geometric drawings of K_n underwent a phase transition by unveiling a close relationship with k-edges. We recall that if P is an n-point set, and $0 \leq k \leq n/2 - 1$, a k-*edge* of P is a line through two points of P leaving exactly k points on one side. A $(\leq k)$-edge is a j-edge with $j \leq k$. The number of k- and $(\leq k)$-edges of P are denoted by $E_k(P)$ and $E_{\leq k}(P)$, respectively. Finally, let $E_{\leq k}(n)$ denote the minimum $E_{\leq k}(P)$, taken over all n-point sets P in general position.

For an n-point set P in the plane in general position, let $\overline{\text{cr}}(P)$ denote the number of crossings in the rectilinear drawing of K_n induced by P. The following was proved independently by Lovász, Wagner, Welzl, and Wesztergombi [26], and by Ábrego and Fernández-Merchant [4]:

$$\overline{\text{cr}}(P) = \sum_{k=0}^{\lfloor n/2\rfloor - 2} (n-2k-3)E_{\leq k}(P) - \frac{3}{4}\binom{n}{3} + (1+(-1)^{n+1})\frac{1}{8}\binom{n}{2}. \tag{2}$$

The relevance of this connection between $\overline{\text{cr}}(P)$ and $E_{\leq k}(P)$ was made evident in both [4] and [26] by proving that

$$E_{\leq k}(n) \geq 3\binom{k+2}{2}, \quad \text{for} \quad 0 \leq k \leq n/2 - 1. \tag{3}$$

Substituting (3) into (2) yields

$$\overline{\text{cr}}(K_n) \geq \frac{3}{8}\binom{n}{4} + \Theta(n^3), \tag{4}$$

thus implying the remarkably improved bound $q_* \geq 3/8 = 0.375$.

We recall that the *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise crossings of edges in a (nonnecessarily geometric) drawing of G in the plane. There are drawings of K_n with exactly $\lambda_n := (1/4)\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor\lfloor (n-2)/2\rfloor \lfloor (n-3)/2\rfloor$ crossings, and it is widely believed that these drawings are crossing-minimal; that is, it is conjectured that $\mathrm{cr}(K_n) = \lambda_n$ for every positive integer n. This conjecture has been verified for $n \leq 12$ [23, 27]. Since $\mathrm{cr}(K_n) \leq \lambda_n$, it follows at once that $\lim_{n\to\infty} \mathrm{cr}(K_n)/\binom{n}{4} \leq 3/8$.

This last upper bound gives an additional significance to (4). With this motivation, Lovász et al. pushed a little further, invoking the following from [34]:

$$E_{\leq k}(n) \geq \binom{n}{2} - n\sqrt{n^2 - 2n - 4k^2 + 4k}. \tag{5}$$

This last bound is better than (3) for $k > 0.4956n$. Using (3) for $k \leq 0.4956n$, and (5) for $k > 0.4956n$, Lovász et al. derived the slightly improved bound $q_* > (3/8) + 10^{-5}$. Although numerically marginal, this improvement is significant because it shows that $\mathrm{cr}(K_n)$ and $\overline{\mathrm{cr}}(K_n)$ differ in the asymptotically relevant term.

5. Lower bounds III: further improvements

Since the key connection (2) was proved in [4] and [26], all subsequent efforts to lower bound $\overline{\mathrm{cr}}(K_n)$ have been focused on finding better estimates for $E_{\leq k}(n)$.

The first improvement was reported in [16], giving a lower bound for $E_{\leq k}(n)$ that is strictly better than (3) for $k > 0.4651n$. The bound given in [16] is in terms of a complicated expression. For our current surveying purposes, it suffices to mention that using this bound Balogh and Salazar proved that $\overline{\mathrm{cr}}(K_n) > 0.37553\binom{n}{4} + \Theta(n^3)$.

Another significant improvement was achieved by Aichholzer, García, Orden, and Ramos [12], who proved that

$$E_{\leq k}(n) \geq 3\binom{k+2}{2} + 3\binom{k+2-\lfloor n/3\rfloor}{2} - \max\{0, (k+1-\lfloor n/3\rfloor)(n-3\lfloor n/3\rfloor)\}. \tag{6}$$

A shorter proof of (6), given in the more general context of pseudolinear drawings was given in [1].

Substituting (6) into (2), one obtains the improved estimate $q_* \geq 41/108 > 0.37962$. Moreover, it is possible to use the bound by Balogh and Salazar [16] in the range $k > 0.4864n$ to obtain the marginally better $q_* > 0.37968$.

The current best lower bound known for q_* is derived using a result recently reported by Ábrego, Cetina, Fernández-Merchant, Leaños, and Salazar [3, 7]. They proved that for every k and n such that $\lceil (4n-11)/9\rceil - 1 \leq k \leq (n-2)/2$,

$$E_{\leq k}(n) \geq u_k(n) \geq \binom{n}{2} - \frac{1}{9}\sqrt{1 - \frac{2k+2}{n}}(5n^2 + 19n - 31). \tag{7}$$

The function u_k is asymptotic to the latter expression and it is better than all previous bounds (including (5) (6), and the bound in [16]) across its full range $\lceil (4n-11)/9\rceil \leq k \leq (n-2)/2$. In addition, Ábrego et al. [3] constructed point-sets achieving equality on (6) for all $k < \lceil (4n-11)/9\rceil$. Using (6) for $k < \lceil (4n-11)/9\rceil$, and (7) for $\lceil (4n-11)/9\rceil \leq k \leq (n-2)/2$, it follows from (2) that $\overline{\mathrm{cr}}(K_n) \geq (277/729)\binom{n}{4} + \Theta(n^3)$, thus implying that $q_* \geq 277/729 > 0.37997$.

6. Upper bounds

The literature on crossing numbers of particular families of graphs is vastly dominated by papers that focus on establishing lower bounds. Most of the time, a natural drawing suggests itself with relatively little effort. When successive attempts to produce better drawings fail, this is seen as plausible evidence that the proposed drawing is indeed optimal. The efforts are then directed in the opposite, and usually remarkably harder, direction: proving nontrivial lower bounds for the crossing numbers of the graphs upon consideration.

The problem of upper bounding the rectilinear crossing number of K_n is a notable exception to this trend. The goal is to describe a way to draw K_n with as few crossings as possible, for arbitrarily large values of n, so as to have at least an educated guess at the asymptotic value $q_* = \lim_{n\to\infty} \overline{\text{cr}}(K_n)/\binom{n}{4}$. Over the years, several strategies to draw K_n with few crossings have been put forward. However, to this day there has not been a clear candidate for an optimal drawing. The only common characteristic is that almost all drawings with few crossings have (or are really close to have) 3-fold symmetry with respect to a point. That is, the underlying point-set P of the drawing is partitioned into three sets (we call them *wings*) of size $n/3$ each, with the property that rotating each wing angles of $2\pi/3$ and $4\pi/3$ around a suitable point generates the other two wings.

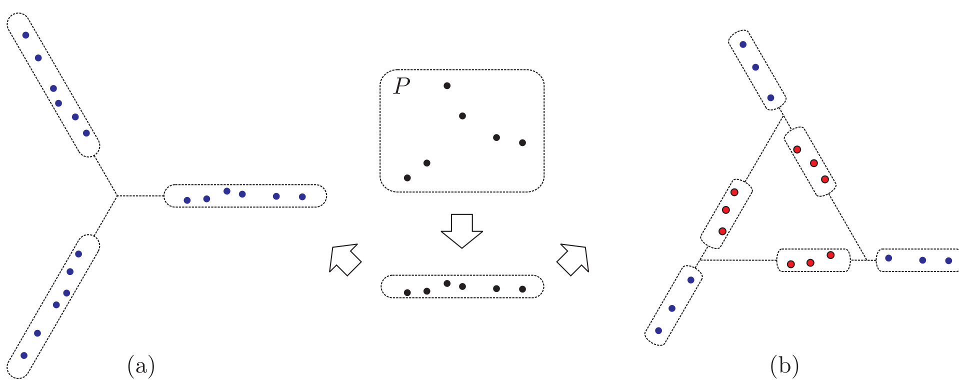


Figure 1. (a) Recursive construction by Singer. (b) Recursive construction by Brodsky et al.

In the early 1970s, Jensen [24] was the first to propose a way to draw K_n for arbitrarily large values of n. His construction gave specific coordinates for $n/3$ points in a wing, and then obtained the remaining two wings by rotating $2\pi/3$ and $4\pi/3$ around the origin. As a result he obtained $q_* \le 7/18 < 0.38889$.

At around the same time, Singer [29] started the trend of recursively constructing drawings of K_n. His idea was to start with a good drawing of $K_{n/3}$, apply an affine transformation to it to make the slope of each of its edges sufficiently close to zero, and then add the $2\pi/3$ and $4\pi/3$ rotations of the resulting drawing to obtain the other two wings. (See Figure 1(a).) This construction shows that

$$\overline{\text{cr}}(K_n) \le 3\overline{\text{cr}}(K_{n/3}) + 3 \cdot \frac{n}{3}\binom{n/3}{3} + 3\binom{n/3}{2}^2.$$

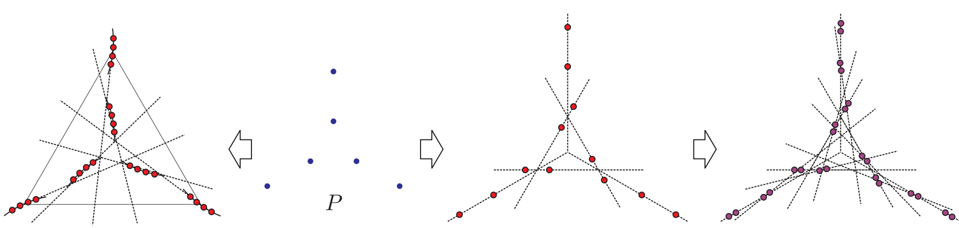
Indeed, the first term consists of the crossings obtained from 4 points in the same wing, the next term counts the crossings from 3 points in one wing and the remaining in one of the other two wings, and the last term counts the crossings from 2 points in one wing and 2 points in another wing. Using $\overline{\mathrm{cr}}(K_3) = 0$ as a starting point, this inequality gives $q_* \leq 5/13 < 0.38462$.

Brodsky, Durocher, and Gethner [19] modified Singer's construction by sliding 3 points in each wing toward the center of rotation as shown in Figure 1(b). Their construction gives $q_* \leq 6467/16848 < 0.38385$.

Aichholzer, Aurenhammer, and Krasser [11] devised a different replacement construction. They started with an underlying set P with an even number of points N. Instead of triplicating P, they replaced every point of P by a cluster of c points on a small arc of circle flat enough so that all lines among these c points leave $N/2$ points of P on one side and $N/2 - 1$ on the other side. (See Figure 2(a).) Letting $n = cN$, their construction gives

$$\overline{\mathrm{cr}}(K_n) \leq \left(\frac{24\overline{\mathrm{cr}}(P) + 3N^3 - 7N^2 + 6N}{N^4} \right) \binom{n}{4} + \Theta(n^3).$$

Using a set P with $N = 36$ points and $\overline{\mathrm{cr}}(P) = 21\,191$ they obtained $q_* < 0.380858$. They further explored using different sizes for each of the clusters, which resulted in an improvement of the latter bound to $q_* < 0.380739$. This method of obtaining lower bounds allowed for improvements by using better initial sets P. Aichholzer and Krasser [15], as part of their computer-assisted search of the crossing numbers $\overline{\mathrm{cr}}(K_n)$ for small values of n, obtained a particular drawing of K_{54} that gives $q_* < 0.380601$.



(a) (b)

FIGURE 2. (a) Replacement construction by Aichholzer et al. (b) Recursive construction by Ábrego and Fernández-Merchant.

Ábrego and Fernández-Merchant started with an underlying set P with an even number of points N. They obtained a new set Q by replacing every point of P by a pair of points close to each other and spanning a line that divides the rest of Q in half. (See Figure 2b). This property of having a *halving-line matching* is not satisfied by an arbitrary point-set P, but fortunately it is satisfied by most of the small sets with optimal crossing number. Moreover, the resulting set Q inherits this property. Thus, if $n = 2^k N$, then iterating this construction k times gives

$$\overline{\mathrm{cr}}(K_n) \leq \left(\frac{24\overline{\mathrm{cr}}(P) + 3N^3 - 7N^2 + (30/7)N}{N^4} \right) \binom{n}{4} + \Theta(n^3). \tag{8}$$

At the time, using the best known drawing of K_{30} (now proved to be optimal) yielded $q_* < 0.380559$. To this date, (8) provides the currently best recursive construction. The restrictions on the base set P were subsequently weakened [2] in the sense that (8) also holds for arbitrary sets P with an odd number of points. Applying this inequality to a drawing of K_{315} with 152 210 640 crossings gives the currently best upper bound: $q_* < \frac{83\,247\,328}{218\,791\,125} < 0.380488$.

To support the belief that the crossing-minimal sets have nearly 3-fold symmetry, Ábrego et al. [2] constructed a 3-fold symmetric set of n points for each n multiple of 3, $n < 100$. (See Figure 3.) Moreover, 3-fold symmetry is inherited from the base set in all recursive constructions mentioned before. In fact, the drawing of K_{315} used as a base set to obtain the best current upper bound has 3-fold symmetry.

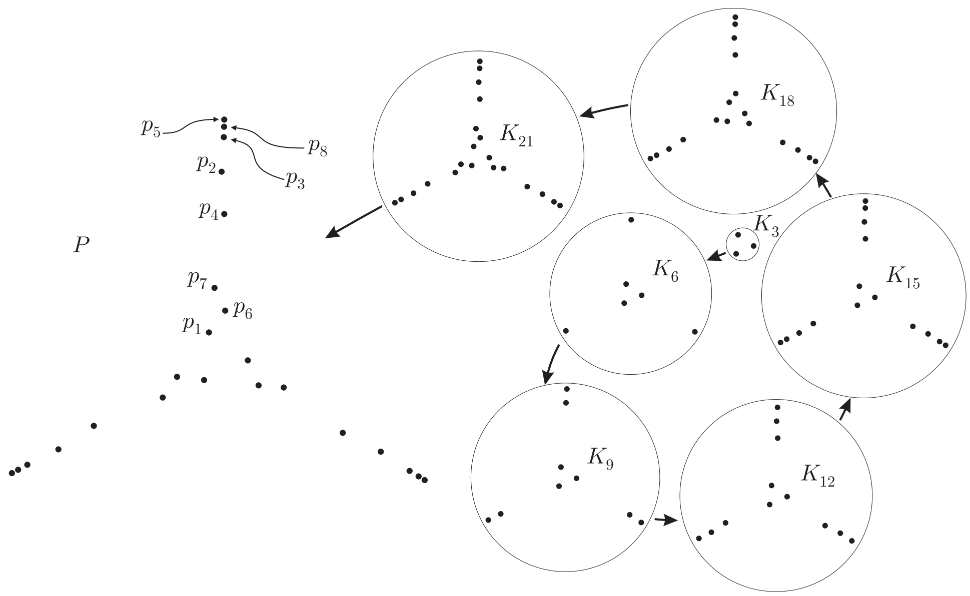


FIGURE 3. The underlying vertex set of an optimal 3-symmetric geometric drawing of K_{24}. This point set contains optimal nested 3-symmetric drawings of $K_{21}, K_{18}, K_{15}, K_{12}, K_9, K_6$, and K_3.

7. Summary

In this section we summarize, for quick reference, the state-of-the-art on $\overline{\text{cr}}(K_n)$ and q_* at the time of writing this survey.

7.1. **Sylvester's Four Point Constant.**

$$0.379972 < \frac{277}{729} \leq q_* \leq \frac{83\,247\,328}{218\,791\,125} < 0.380488. \tag{9}$$

The lower and upper bounds in (9) are derived in [3] (see also [7]) and [2], respectively.

7.2. **Exact values of $\overline{\text{cr}}(K_n)$.** The exact value of $\overline{\text{cr}}(K_n)$ is known for $n \leq 27$ and for $n = 30$ (see Table 1).

For $n \leq 27$, the lower bound for $\overline{\text{cr}}(K_n)$ is derived in [3] (see also [7]). The bound $\overline{\text{cr}}(K_{30}) \geq 9726$ is proved in [20]. In all cases, the upper bounds were obtained by Aichholzer [8].

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\overline{\text{cr}}(K_n)$	1	3	9	19	36	62	102	153	229	324	447	603	798	1029

n	19	20	21	22	23	24	25	26	27	30
$\overline{\text{cr}}(K_n)$	1318	1657	2055	2528	3077	3699	4430	5250	6180	9726

TABLE 1. Exact rectilinear crossing numbers known.

8. Further thoughts and future research

Since the introduction of (2) in [4] and [26], all the progress achieved on lower bounding q_* has been contingent on the derivation of improved bounds for $E_{\leq k}(n)$.

Although it may seem natural to expect the continuation of this trend, there is some evidence that suggests that this approach alone will not lead to the correct value of q_*. The reasons behind our caution lie on our own investigations of sets that minimize the number of $(\leq k)$-edges. So far it has been possible to construct n-point sets that simultaneously minimize $E_{\leq k}(n)$ for al k up to a certain value. It is not difficult to construct an n-point set that simultaneously achieves equality in (3) for every k, $0 \leq k \leq n/3$, and arbitrary n (along this discussion we assume that n is a multiple of 3). To construct a similar set minimizing $E_{\leq k}(n)$ for a larger range of values of k is notably harder. Aichholzer, García, Orden, and Ramos [13] constructed an n-point set that simultaneously achieves equality in (6) for every k, $0 \leq k \leq \lfloor \frac{5n}{12} \rfloor - 1$ and arbitrary n. A different type of construction was used in [3] to simultaneously show that (6) is tight for every k, $0 \leq k \leq 4n/9 - 1$. However, this construction is far from crossing optimal due to a dramatic increase on the number of $(\leq k)$-sets when $k \geq 4n/9$, and avoiding this seems impossible. That is, insisting on simultaneously minimizing $E_{\leq k}(n)$ for all k, for k as large as possible, seems to actually increase the crossing number of the point sets under consideration. In view of this, a new paradigm might be in order. It seems not only possible, but very likely, that the crossing-minimal drawings of K_n for large values of n are attained by point sets that are not even close to minimizing $E_{\leq k}$ for every $k < (4n/9) - 1$. A proper understanding of this intriguing behavior seems out of our reach at the present time.

Although (2) validates the efforts to lower bound $\overline{\text{cr}}(K_n)$ via lower bounding $E_{\leq k}(n)$, our previous remarks suggest that, no matter how fine the estimates, this may not suffice in order to determine q_*. It is quite conceivable that the (exact or asymptotic) value of $E_{\leq k}(n)$ be known for every k, and still the estimate for $\overline{\text{cr}}(K_n)$ obtained from plugging this into (6) does not correspond to the correct (at least asymptotic) value of $\overline{\text{cr}}(K_n)$.

The outlook from the upper bounds front is also unclear. We might all be just one clever idea away from a breakthrough construction that yields (at least asymptotically) crossing-minimal geometrical drawings of K_n. Using best-case heuristics, it can be shown that any recursive construction for large N, where each point is replaced by a small cluster of points of the same size, can yield at best a bound of the form

$$\overline{\mathrm{cr}}(K_n) \leq \left(\frac{24\overline{\mathrm{cr}}(P) + 3N^3 - 7N^2 + 4N}{N^4}\right)\binom{n}{4} + \Theta(n^3).$$

The improvement using the best drawing of K_{315} would be less than 10^{-8}.

For all these reasons, we are inclined to think that there is more potential to close the gap from below than from above; that is, we believe that q_* is closer to the current best upper bound than to the current best lower bound.

To end on an optimistic note, there is one more promising observation. Besides 3-fold symmetry, the currently best known constructions (including those presented in [2]) share another property called 3-decomposability. A set P is called *3-decomposable* if there exists a balanced partition of P into three parts A, B, and C and a triangle T enclosing P such that the orthogonal projections of P onto the sides of T show A between B and C on one side, B between A and C on another side, and C between A and B on the third side. As for 3-fold symmetry, 3-decomposability is inherited from a base set in all recursive constructions mentioned before. Ábrego et al. [2] conjectured that all crossing-minimal sets are 3-decomposable. If this conjecture happens to be true, then the lower bound for q_* would be improved to $(2/27)(15 - \pi^2) > 0.380029$ as proved in [2].

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